Pythagorean Theorem with Hippocrates’ Lunes

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Pythagorean Theorem with Hippocrates’ Lunes

Abstract
Is it possible to place a lune on the hypotenuse of a right triangle whose area is equal to the sum of the areas of the other two lunes placed on the legs of the triangle? In this article, I use dynamic geometry software and spreadsheets in an attempt to answer this question along with the conditions satisfying the existence of such a lune. This in the classroom article also offers a method of investigating a trigonometric equation involving two variables using spreadsheets and dynamic geometry snapshots that are presented in a manner that complements the analytic and the visual approaches. I conclude by reinforcing the idea that the Pythagorean Theorem is indeed a relationship of areas, with or without the restriction that the lunes placed on the sides of a right triangle be similar.

Keywords
Hippocrates of Chios, quadrature, lunes, dynamic geometry software, Pythagorean Theorem, spreadsheets

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Hippocrates’ Lunes and the Pythagorean Theorem:
A Spreadsheet Approach for Solving the Lune Condition

Hippocrates of Chios (c. 440 B.C.) successfully squared certain lunes, plane figures bounded by two circular arcs. Books III and IV, which details Euclid’s work on the properties of circles, greatly influenced the development of Greek science and astronomy; “they became part of the Greek mathematician’s toolbox for solving other problems... Hippocrates used results on circles in his quadrature of lunes” (Katz, 2009, p.66). “One of the earliest Greek mathematicians to attempt to treat the problem of the “quadrature of the circle” in pure geometric form, with the specific restriction that only compass and straightedge should be used, was Hippocrates of Chios’” (NCTM, 1989, p.150). Hippocrates showed that certain lunes could be “squared,” that is, the areas of such lunes would be equivalent to the areas of certain polygons (e.g., triangles, rectangles, etc.) (Midonick, 1965; Van Der Waerden, 1963). His proofs relied on a fundamental principle that intertwined similarity and area conservation: “the areas of circles are to one another as the squares on their diameters, a fact evidently known to the Babylonian scribes” (Katz, 2009, p.41). For example, if $\triangle AOC$ is a quarter circular sector and another semicircle is constructed in such a way that the diameter of the semicircle coincides with chord $AC$, then it can be shown that triangle $AOC$ and lune $AFCE$ (in red) bounded by the semicircle and the quarter circular sector have equal areas (Fig.1)

![Figure 1: Lune formed on a quadrant of a circle](image)

In what follows, I investigate the lunes formed on the three sides of a right triangle.
I. Lunes Formed on the Three Sides of a Right Triangle

Inscribing a right triangle $ABC$ in a semicircle $ACB$, followed by the construction of lunes $AFCE$ and $BHCG$ on both legs (Fig.2), it can be shown that the conjectures (i) the area of lune $AFCE$ is equal to the area of triangle $AOC$; (ii) the area of lune $BHCG$ is equal to the area of triangle $BOC$ are false. However, calculations on Geometer’s Sketchpad show that the sum of the areas of the two lunes is equal to two times the area of $AOC$ (Fig.2). It is also worth noting that triangles $AOC$ and $BOC$ are both isosceles triangles (with legs as radii of the semicircle $ACB$) with the same area because they have the same length of base and height.

**Proof:** Let $AB = c$, $BC = a$, $AC = b$. (i) The area of semicircle $AFC$ is equal to $\pi \cdot \left(\frac{b}{2}\right)^2 = \frac{\pi b^2}{4}$; (ii) the area of semicircle $BHC$ is equal to $\frac{\pi a^2}{4}$; and (iii) the area of right triangle $ABC$ is equal to $\frac{ab}{2}$. The sum of these three areas must be equal to the area of the big semicircle $ABC$ (which is $\frac{\pi c^2}{4}$) plus the sum of the areas of lunes $AFCE$ and $BHCG$. It can then be deduced that the sum of the areas of lunes $AFCE$ and $BHCG$ is equal to $\frac{\pi b^2}{4} + \frac{\pi a^2}{4} + \frac{ab}{2} - \frac{\pi c^2}{4} = \frac{\pi}{4} (b^2 + a^2 - c^2) + \frac{ab}{2}$, which implies that the sum of the areas of the two lunes is equal to the sum of the areas of triangles $AOC$ and $BOC$, which is equal to the area of the original right triangle $ABC$ (Fig.3).

![Figure 2: Lunes formed on the legs of a right triangle](image-url)
Problem Statement: Is it possible to place a lune (on the hypotenuse of triangle $ABC$) whose area is equal to the sum of the areas of the other two lunes (on the legs)? This problem has not been investigated before. Dynamic geometry software may be used in an attempt to answer this question along with the conditions satisfying the existence of such a lune.

Similarity Arguments: First of all, similarity arguments do not apply because the circular segments $AEC$ and $BGC$ are not similar: They are similar only when the right triangle $ABC$ is isosceles, in which case the circular arc sectors containing these segments are the same fraction (i.e., one fourth) of their circles. In general, as evident by Fig.2, the circular segments $AEC$ and $BGC$ are not similar.

Area Conservation Principle: The area of circular segment $AEC$ is equal to $\frac{1}{2} \pi \left( \frac{b}{2} \right)^2 - \frac{ab}{4}$; the area of circular segment $BGC$ is equal to $\frac{1}{2} \pi \left( \frac{a}{2} \right)^2 - \frac{ab}{4}$. The sum of the areas of these circular segments is therefore equal to $\frac{1}{2} \pi \left( \frac{a^2+b^2}{4} \right) - \frac{ab}{2}$, which can be simplified as $\frac{1}{2} \pi \left( \frac{c^2}{4} \right) - \frac{ab}{2}$ by using the Pythagorean Theorem $a^2 + b^2 = c^2$ for triangle $ABC$ (Fig.2). Namely the desired circular segment to be placed on side $AB$ (the hypotenuse) should be of area $\frac{1}{2} \pi \left( \frac{c}{2} \right)^2 - \frac{ab}{2}$, which is equivalent to the area of semicircle $ABC$ of diameter $c$ minus the area of right triangle $ABC$.

Constructing a Circular Segment on the Hypotenuse: How to determine the center $M$ of a circular segment with known area and chord length? How far should that center be from either endpoint of chord $AB$? Figure 4 illustrates the circular segment $ALB$ on the hypotenuse $AB$ of triangle $ABC$ with center $M$ that is located on the perpendicular bisector of the hypotenuse $AB$. $\theta$ is the radian measure of the central angle $\angle AMB$ of the
corresponding arc sector. \( R = AM = LM = BM \) is the radius of the same arc sector. By proportional reasoning, the area of the arc sector is \( \frac{\theta R^2}{2} \), where \( R \) is the radius of the arc sector. Using the Heron’s formula \( \sqrt{s(s - R)(s - R)(s - c)} \) with semiperimeter \( s = \frac{1}{2} (R + R + c) \), the area of the isosceles triangle \( AMB \) contained in the arc sector is given by \( \frac{c}{2} \sqrt{R^2 - \frac{c^2}{4}} \). The area of the sought circular segment \( ALB \) is therefore \( \frac{\theta R^2}{2} - \frac{c}{2} \sqrt{R^2 - \frac{c^2}{4}} \).

Expressing \( \theta \) in terms of \( c \) and \( R \), the last expression becomes

\[
R^2 \sin^{-1} \left( \frac{c}{2R} \right) - \frac{c}{2} \sqrt{R^2 - \frac{c^2}{4}}
\]

Setting this equal to \( \frac{1}{2} \pi \left( \frac{c}{2} \right)^2 - \frac{ab}{2} \), the equation

\[
R^2 \sin^{-1} \left( \frac{c}{2R} \right) - \frac{c}{2} \sqrt{R^2 - \frac{c^2}{4}} = \frac{1}{2} \pi \left( \frac{c}{2} \right)^2 - \frac{ab}{2}
\]

is obtained. This equation implies that \( R \) can be obtained once the lengths of sides \( a, b, c \) are known.

\[
m\angle AMB = \theta, \ m\angle AMO = m\angle BMO = \alpha
\]

\[
\alpha = \frac{\theta}{2} = \arcsin \left( \frac{c}{2R} \right)
\]

\[
s = \frac{1}{2} (R + R + c) = R + \frac{c}{2}
\]

Figure 4: Attempts to locate the center of the circular segment with hypotenuse \( AB \) as chord

**Illustrating the Equation**: The last equation can be illustrated in a dynamic geometry software, such as Geometer’s Sketchpad (Fig.5). The calculations also verify the above derived equation

\[
R^2 \sin^{-1} \left( \frac{c}{2R} \right) - \frac{c}{2} \sqrt{R^2 - \frac{c^2}{4}} = \frac{1}{2} \pi \left( \frac{c}{2} \right)^2 - \frac{ab}{2}
\]

It is also worth noting that not only is it possible to place a circular segment \( AKB \) on the hypotenuse \( AB \) with the sought conditions, it is possible to place a lune \( AKBL \) with the same sought conditions as well.
Figure 5: Verifying the equation

The Lune Condition: Using \( \theta = \frac{\pi}{2} = \sin^{-1} \frac{c}{2R} \), the lune condition may be rewritten as

$$\alpha R^2 - \frac{c}{2} R \cos \alpha = \frac{1}{2} \pi \left( \frac{c}{2} \right)^2 - \frac{ab}{2}.$$ 

Multiplying both sides by \( \frac{4}{c^2} \) gives

$$\alpha \left( \frac{2R}{c} \right)^2 - 2 \frac{ab}{c^2} = \frac{\pi}{2} - 2 \frac{ab}{c^2} \Rightarrow \alpha \csc^2 \alpha - \cot \alpha = \frac{\pi}{2} - 2 \frac{ab}{c^2} \Rightarrow \alpha \csc^2 \alpha - \cot \alpha - \frac{\pi}{2} + 2 \frac{ab}{c^2} = 0.$$ 

Defining \( \beta = m \angle BAC \), the last equation can be written in terms of \( \alpha \) and \( \beta \) only: \( \alpha \csc^2 \alpha - \cot \alpha - \frac{\pi}{2} + 2 \sin \beta \cos \beta = 0 \Rightarrow \alpha \csc^2 \alpha - \cot \alpha - \frac{\pi}{2} + \sin 2\beta = 0 \), which is the trigonometric version of the lune condition.

II. Solving the Trigonometric Equation involving \( \alpha \) and \( \beta \)

In this section, the conditions satisfying the existence of this lune, in particular, the trigonometric equation involving two variables will be explored using the spreadsheets feature of GeoGebra dynamic software.

Defining Quantities on Spreadsheets: Column B will be used to enter angle measures (in radians) for \( \beta \), starting with \( B1 = \frac{\pi}{180} \), \( B2 = \frac{2\pi}{180} \), \( B3 = \frac{3\pi}{180} \), up to B45. Column C will be used to determine the value of \( \frac{\pi}{2} - \sin 2\beta \) for each \( \beta \) defined in Column B.

Graphics view will be used to graph the function \( f(\alpha) = \alpha \csc^2 \alpha - \cot \alpha \) within the domain \([0, \frac{\pi}{2}]\) using the syntax \( \text{Function}[x \ (\csc(x))^2 - \cot(x), 0, \pi/2] \). The range of this function is identified from the graph as \([0, \frac{\pi}{2}]\). Column D will be used to define the horizontal lines via the syntax \( D1 = \text{Function}[C1, 0, \pi/2] \) and dragging down to D45. These horizontal lines appear on the Graphics view as well (Figs.6-7).
Column E will be used to define the abscissas (i.e., the values of angle $\alpha$) of the intersections of the horizontal lines D1:D45 with the function $f$ via the syntax $E1 = x(\text{Intersect}[f, D1, 0, \pi/2])$ and dragging down to E45. That way, all $\alpha$ (values in radians) obtained in E1:E45 making the equation $\alpha \csc^2 \alpha - \cot \alpha = \frac{\pi}{2} - \sin 2\beta$ true for each $\beta$ defined in B1:B45. Column A will be used to enter the values of angle $\alpha$ (in degrees for comparison purposes) via the syntax $A1 = E1 (180/\pi)$ and dragging down to A45. Column F can be used to record any function of $\alpha$ and $\beta$, such as the sum $\alpha \csc^2 \alpha + \cot \alpha$ in degrees. Figure 7 illustrates all spreadsheet data on Columns A:F.

Figure 6: Horizontal lines from Column D

Figure 7: Spreadsheet data in Columns A:F
Analysis of the Spreadsheet Data: It is worth noting that when $\beta = 45^\circ$, $\alpha = 45^\circ$ too. This particular case is prone to several observations.

1) This means that if the original right triangle $ABC$ is isosceles, then points $C$ and $M$ would coincide; namely that triangles $ABC$ and $AMB$ would coincide too.

2) All three circular segments forming on the sides of triangle $ABC$ would be similar. In particular, the segments $AEC$ and $BGC$ forming on the legs would be congruent (Fig. 8a).

3) All three lunes forming on the sides of triangle $ABC$ would be similar as well. In particular, the lunes $AFCE$ and $BHCG$ forming on the legs of triangle $ABC$ would be congruent (Fig. 8b).

Observations Regarding the Sum $\alpha + \beta$: For a given $\beta$, the maximum value of the sum $\alpha + \beta$ is $90^\circ$, which occurs for $\beta = 45^\circ$. The minimum value of the sum $\alpha + \beta$ occurs for $\beta \approx 31^\circ$. Is it possible to bound $\beta$ so that the sum $\alpha + \beta$ is minimum? Redefining $B1 = 30.917 \left( \frac{\pi}{180} \right), B2 = B1 + 0.00001 \left( \frac{\pi}{180} \right)$, etc., it is possible to regenerate the spreadsheets data with 2000 rows for $\beta \in [30.917^\circ, 30.919^\circ]$ to obtain the desired interval $\alpha + \beta \in [83.4615682475^\circ, 83.4615682477^\circ]$ for the sum (Fig. 9).
Figure 9: Searching for a bound for the minimum value of the sum $\alpha + \beta$

Locus the Sum $+ \beta$: Defining a slider for $\beta \in [0^\circ, 45^\circ]$, it is possible to obtain the locus of the sum $\alpha + \beta$ by introducing the locus point $P = (\beta, \alpha + \beta)$ where the values of $\alpha$ and $\beta$ are obtained from the spreadsheets data (Fig.10). In Figure 10, the value of angle $\alpha$ (shown in radians) is obtained from the graph of $f$ by the intersection of the horizontal line (obtained from Column D) and the graph of $f$. The locus curve indicates that the sum value is minimum for $\beta \approx 30.9^\circ$, in agreement with the spreadsheet data.

III. Conclusion

It therefore is possible to place a lune on the hypotenuse of a right triangle whose area is equal to the sum of the areas of the other two lunes placed on the legs of the triangle if and only if the equation $\alpha \csc^2 \alpha - \cot \alpha = \frac{\pi}{2} - \sin 2\beta$ holds (the lune condition). The
Pythagorean Theorem is indeed a relationship of areas, with or without the restriction that the lunes placed on the sides of a right triangle be similar.

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