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1 Abstract

We describe a mathematical model and simulation study for Jackpot Keno, as implemented by Jupiters Network Gaming (JNG) in the Australian state of Queensland, and as controlled by the Queensland Office of Gaming Regulation (QOGR) [7]. The recurrences for the house net hold are derived and it is seen that these are piecewise linear with a ternary domain split, and further, the split points are stochastic in nature. Since this structure is intractable [1], estimation of house net hold obtained through an appropriately designed simulator using a random number generator with desirable properties is described.
Since the model and simulation naturally derives hold given payscale, but JNG and QOGR require payscale given hold, an inverse problem was required to be solved. This required development of a special algorithm, which may be described as a stochastic binary search.
Experimental results are presented, in which the simulator is used to determine jackpot payscales so as to satisfy legal requirements of approximately 75% of net revenue returned to the players, i.e., 25% net hold for the house (JNG). Details of the algorithm use to solve this problem are presented here, and notwithstanding the stochastic nature of the simulation, convergence to a specified hold for the inverse problem has been achieved to within 0.1% in all cases of interest to date.

Keywords: Jackpot, Keno, recurrence, simulation.

2 Introduction

The Chinese lottery game, Keno, has been played for many centuries, traditionally with very simple implements [2]. Of course, modern 21st century versions of the game are computerised and available on many Internet sites. In Australia, various casinos offer their versions of the game. The basic game, the so-called *Regular Keno*, is very simple, and all commercial versions known to the authors implement essentially the same simple binomial structure. It is the jackpot component of the game where implementations differ significantly. This paper is concerned with the modelling and analysis of a networked implementation of jackpot Keno in the Australian state of Queensland, available in hotels, clubs, totalisator agencies (TABs) and other similar establishments. Although *Regular Keno* has a simple binomial structure, that of its jackpot version, as implemented in Australia by Jupiter's Network Gaming (JNG), has a much more intricate mathematical structure. Modelling of JNG's version of the jackpot game leads to stochastic recurrence relations, and analysis of the model requires the solution of intractable sets of difference equations. For the remainder of this paper, the term *Keno* refers to the game as implemented by JNG in Queensland, Australia, although at the time of writing, JNG has just acquired the rights to expand its Keno game into New South Wales, the most populous Australian state.

Versions of this work have been presented by the second author at the 34th Applied Mathematics Conference (ANZIAM 98), Australian Mathematical Society, Greenmount, February 7-11, 1998 [9], and at the Fourth International Conference on Difference Equations and their Applications (ICDEA 98) in Poznan, Poland, in August 1998 [10].

2.1 Keno basics

In every Keno game, there are 80 balls, numbered 1 to 80. Every three minutes, a new game of Keno is played, in which 20 of the 80 balls are drawn at random. Bets are accepted for the three minutes, after which the game is closed and the balls drawn. As each ball is drawn, its number is displayed in an 8 by 10 table on television monitors statewide in participating establishments. To play the game, players mark numbers by pencil on a betting ticket, which is simply an 8 by 10 matrix of the numbers 1 to 80, and pay their \$1. The number of values, or *spots*, marked on the ticket determines the *bet-type*. In the JNG implementation, currently-available bet-types are $1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 15, 20, 40$ spots, and each of these bet-types has an associated payscale. A payscale is merely a table giving the amount won for the minimum \$1 bet for each of the possible catches, r, bounded as shown by inequality 2. Table 1 shows a typical $7 - spot$ payscale.

A player may also choose to play by allowing the house to pseudo-randomly select numbers on a Keno ticket.

Consider bet-type k, in which a player marks k values on a ticket. We have $k \in K$, where K is the set of available bet types, given by eq (1).

$$K = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 15, 20, 40\} \tag{1}$$

The industry jargon for the number of player-chosen values matching those in the drawn set of 20 is the *catch*. We use the symbol r to represent the catch in any particular $k - spot$ bet. Thus, we have:

$$0 \leq r \leq \min(k, 20) \tag{2}$$

Table 1: Payscale and expected payouts for regular 7-spot game

r	$\Pr(\text{catch} = r)$	**Payout ($)**	**Expected Payout ($)**
0	0.12157	0	0.00
1	0.31519	0	0.00
2	0.32665	0	0.00
3	0.17499	1	0.17
4	0.05219	3	0.16
5	0.00864	15	0.13
6	0.00073	95	0.07
7	0.00002	3,000	0.07
		Total	0.60

For some bet types (see section 4.1), a fixed portion of each player dollar is channeled into several special (k, r) *jackpot meters*. In addition to regular Keno prizes described above, a jackpot prize may be won. At present, this occurs if and only if $r = k$, i.e., the numbers on a player's ticket are all drawn. Unless a (k, r) jackpot is won, the relevant (k, r) jackpot meter continues to accumulate, as further games are played, until a specified limit is reached. These processes are described in detail in the sections to follow.

3 Regular Keno

The probability, Pr, of obtaining catch r in a $k - spot$ regular game of Keno is given by the following function of binomial coefficients.

$$\Pr = \binom{20}{r} \times \binom{60}{k-r} \div \binom{80}{k} \tag{3}$$

For example, applying this formula to a $7-spot$ regular game ($k = 7$), we obtain the probabilities and expectations shown in Table 1.

In the table, while the actual payout figures are exact, the probability values and expected payouts are rounded to 5D and 2D respectively. As usual, the expected payout values are the product of the actual payouts and probabilities, and the sum of $0.60 represents the expected return to the player after an investment of $1 in a regular $7 - spot$ game. It may be thought from this that the house is therefore making 40 cents in the dollar, i.e., 40% profit or *hold*. This conclusion, while apparently entirely reasonable, would be invalid, since it assumes that all player revenue is channeled into the regular game. In fact, for this example, only 80 cents out of every dollar goes to finance the regular game; the remaining 20 cents being used to augment the jackpot meter(s), which are discussed in the sections to follow. It is the mathematical modelling and analysis of this jackpot facet of the game and the impact on long-term hold that is our concern in the remainder of the paper. In the light of the actual revenue input to the regular 7-spot game, it is seen that the return to players is 60 cents out of 80, or in other words, the *return to player* is 75%, and its complement, *the hold* is 25%, which is just below the legal limit of 25.19%.

A related piece of work, conducted by the present authors, concerned statistical analysis of actual Keno draws [5], [6]. This work was required in order to ensure an acceptable level of randomness, and used several standard tests taken from Knuth [3]. As noted, the relevant government body requiring these analyses is the Queensland Office of Gaming Regulation (QOGR). Its world wide web page [7] gives a very brief overview of JNG's networked Keno game in Queensland.

4 Jackpot Keno

In the following sections, we explore the mathematical structure of Jackpot Keno, as implemented by JNG, and describe the development of software to compute final house percentage (hold). An acceptable mathematical analysis is required for two reasons: to satisfy government requirements of correct hold derived from a sound mathematical model, and secondly, to determine a set of payscales so that the overall, long-term hold does not exceed 25.19% (legal limit), but which approaches this bound as closely as possible. In addition to being satisfactory to both QOGR and JNG in these senses, the payscales must also be attractive to Keno players.

The main aim of the study described in this paper was to satisfy a request by JNG to develop, for their networked Keno game, the sound mathematical foundation, necessary, but not sufficient for government approval of the system. The model developed naturally provides the long-term hold, given the payscale. While this is useful, it is of greater interest to solve the inverse problem, viz., to compute the payscale necessary to achieve a given hold. A simulator was created by the authors to satisfy both of these requirements, and was developed using Borland's 32-bit Delphi, running under Microsoft Windows NT 4.0, and Microsoft Windows 2000 Professional. The inverse problem is solved using essentially a binary search algorithm, although there is some delicacy in ensuring the convergence of this method, as the whole simulation is of course, stochastic, and non-reproducible. On the average, over 100,000 simulated jackpots, convergence was achieved to within 0.1% in all cases of interest, to date. The algorithm used to solve the inverse problem is briefly described in section 11 and experimental results are presented in section 12.

Some recent information about JNG Keno may be found at [12]. We quote: *"The Keno system was developed by Jupiters Limited and approved by the Queensland Office of Gaming Regulation in June 1997. Since then, Jupiters Keno has been installed in 4 casinos, over 550 clubs and hotels and is available through 260 off-track betting outlets managed by TAB Queensland."* The August 2001 edition of JNG's Kenotes bulletin [14] also has: *"Jupiters Keno has now entered its fifth year of operation in Queensland. In that time several hundred millions of* *dollars has been won by lucky Queenslanders in clubs, hotels, TABs and casinos statewide."* Notably, this publication does not mention the total revenue put though the game over the past four years, however, some information on total operating revenue (including Keno) may be gleaned from [13].

4.1 Player contribution

As noted in section 2.1, the minimum player investment in a Keno game is one dollar. If the player chooses a bet-type, k, having a jackpot component (presently such $k \in \{7, 8, 9, 10\}$; see section 5.1), then a component, $y(k)$, of that dollar is used to increment the jackpot meters for that bet-type. The remainder, $1 - y(k)$ goes to a separate pool to support the regular game. The value $y(k) = 0.2$ (twenty cents) has been used for all jackpot bet-types, but this may soon change to ten cents, as JNG wishes to maintain consistency with parameters used in the Australian state of New South Wales.

5 Secondary keno jackpots and auxiliary jackpot reset meters

5.1 Purpose of the meters

Jackpot meters are discussed in detail in the sections to follow. In summary, for each jackpot bet-type, k, there is a so-called *visible* meter for each possible catch, r, and an *auxiliary* meter for each possible catch, where $0 \leq r \leq k$. Thus, for $k = 7$ (7-spot bet-type), we would have 8 visible meters and 8 auxiliary meters. The visible meters accumulate revenue, being a fixed percentage of player contributions, and their levels are displayed from time to time on Keno monitors in participating establishments. A visible meter simply shows players what a jackpot payout would be if it were won on the current game. By contrast, auxiliary meter contents are never shown to players. Such meters exist only as reserves, so that when a jackpot is won, the relevant visible meter may be replenished from the corresponding auxiliary meter. In some implementations of Jackpot Keno, meters have no upper limit, e.g. New South Wales "Club Keno", however in the JNG implementation, upper limits are imposed.

5.2 Secondary meters

Consider the k-spot jackpot game. Instead of a single jackpot meter, as is usual in many implementations of jackpot Keno, we have many meters. These additional jackpots are to be incremented from the same player contributions as existing Jackpots. Firstly, a new jackpot meter is created for each catch r, $(0 \leq r \leq k - 1)$. These are termed *secondary meters* and are labelled $(k, 0), (k, 1), (k, 2), ..., (k, r), ..., (k, k - 1)$ respectively. We term the original k-spot jackpot meter the *primary jackpot meter,* which by an obvious extension of the above, we label (k, k), and allow $0 \leq r \leq k$. The purpose of secondary meters is to allow for the possibility of jackpot payouts for a less than perfect catch, i.e., for $r < k$. For secondary meter (k, r), let the fraction of the total player contribution to the k-spot jackpot game augmenting this meter be $\lambda(k, r)$, with $\lambda(k, k)$ representing the proportion augmenting the primary meter. Thus, for each k, we have the relationship of equation (4).

$$\sum_{r=0}^{k} \lambda(k, r) = 1 \tag{4}$$

It should be noted that, at present, JNG has $\lambda(k, k) = 1$ for $k \in \{7, 8, 9, 10\}$, while $\lambda(k, r) = 0$ for all other (k, r) pairs. This implies that secondary jackpots are not yet in operation, i.e., that jackpots exist only for perfect catch $(r = k)$, and also that the only "jackpot bet-types" are $7, 8, 9, 10$.
We now define two quantities used in the following sections. $P_{\min}(k, r)$ and $P_{\max}(k, r)$ are respectively the minimum and maximum payouts for catch r in the (k, r) jackpot game.

5.3 Auxiliary meters

In addition to the secondary meters described in the previous section, for each k, and for each r, $(0 \leq r \leq k)$, we have an *auxiliary* meter. In order to distinguish these from the original jackpot meters, we refer to the original (k, r) meter as the *visible* meter. That is to say, for every k, each of the (visible) jackpot meters $0, 1, 2, 3, ..., k$ has a corresponding auxiliary meter. The purpose of the auxiliary meter is to allow reset of the subsequent visible meter to a more acceptable level than its reset, $P_{\min}(k, r)$, after a jackpot is won. If the auxiliary meter contents is less than $P_{\min}(k, r)$, then the house must make up the difference from its own funds.
For a given k and r, the visible meter increases in the normal manner until it reaches $P_{\max}(k, r)$, assuming the (k, r) jackpot has not been won.
No revenue is channeled into an auxiliary meter until the corresponding visible meter has reached its limit $P_{\max}(k, r)$. Once this happens, and provided the (k, r) jackpot has still not been won, then the same proportion, $\lambda(k, r)$, of further player contributions is now channelled into the (k, r) *auxiliary* meter. The auxiliary meter (k, r) is now allowed to grow until it reaches its ceiling, $\sigma P_{\max}(k, r)$, or if a (k, r) jackpot is won, where $\sigma \geq 1$ is a pre-specified constant. The value of σ is prescribed by a regulatory authority, and is discussed in the next section.
If the ceiling, $\sigma P_{\max}(k, r)$, is reached, all further increments are profit to the house; thus, the value of the (k, r) auxiliary meter never exceeds $\sigma P_{\max}(k, r)$. Note that a (k, r) auxiliary meter need not start at zero, but, after a jackpot $\#x$ has just been won, its contents will be used to boost the new visible meter, and any remainder will form the auxiliary meter's initial value for jackpot $\#(x + 1)$. After a jackpot has been won, as much as possible from the (k, r) auxiliary meter, will be used to boost the new visible meter, but never such that the contents of the visible meter exceed $P_{\max}(k, r)$. Suppose (k, r) jackpot $\#x$ has just been

won. Then, the initial value, or reset, of the visible meter for jackpot $\#(x+1)$ will be the sum of $P_{\min}(k,r)$ and the contents of the auxiliary meter, up to a limit of $P_{\max}(k,r)$. Note that auxiliary meter (k,r) is only incremented by the appropriate proportion of player contributions once the pool corresponding to visible meter (k,r) has reached its maximum payout level, $P_{\max}(k,r)$.

We now define $t_x(k,r)$ to be the total number of player contributions to the (k,r) jackpot meter when the xth jackpot is won. Note that $t_x(k,r)$ starts at zero at the very beginning of the first game after (k,r) jackpot $\#(x-1)$ has been won.

We also define $v_x(k,r,t_x)$ to be the dollar amount of the visible jackpot pool after $t_x(k,r)$ contributions for the rth (visible) meter of the xth successive instance of k-spot jackpot game; the corresponding pool value of the auxiliary meter will be denoted $a_x(k,r,t_x)$. For convenience and ease of reading, in the following analysis we mask the dependence of $t_x(k,r)$ on (k,r) and simply write t_x. If we denote the minimum and unit player contribution to k-spot jackpot game by the parameter $y(k)$, then for k-spot jackpot game $\#x$, the total revenue after t_x player contributions is given by the expression $y(k)t_x(k,r)$. Note that at present, we do not consider variable rates of incrementation, i.e., all rates are piecewise constant.

5.4 Some comments on the parameter σ

The parameter σ is a real number whose value must not be less than 1, by decree of QOGR. This means that the maximum contents of any auxiliary meter must be at least that of the corresponding visible meter. It is worth noting here that the value of σ has been a matter of some considerable debate between JNG and QOGR. At the time of writing, its value is $\sigma = 2$, but JNG is seeking to reduce it to $\sigma = 1$. Inexplicably, the originally specified value by QOGR was $\sigma = \infty$. Since no component of the player contribution is removed by JNG before passing to the meters (as might be the case with a totalisator system), this gave rise to a situation where JNG could not remove revenue after the relevant auxiliary meter(s) had filled, i.e., never! This was clearly an intolerable situation for them.

6 Quantities governing secondary and auxiliary reset meters

6.1 Non-dimensional approach

The analysis is greatly simplified by expressing most quantities in terms of non-dimensional parameters. Essentially, this corresponds to working with quantities which are expressed directly in terms of number of player contributions rather than dollar values. The most important of these are the *non-dimensional visible meter contents* $V_x(k,r,t_x)$ and the *non-dimensional auxiliary meter contents* $A_x(k,r,t_x)$, defined respectively by eqs (5) and (6).

$$V_x(k,r,t_x) = \frac{v_x(k,r,t_x)}{y(k)\lambda(k,r)} \tag{5}$$

$$A_x(k,r,t_x) = \frac{a_x(k,r,t_x)}{y(k)\lambda(k,r)} \tag{6}$$

Several other significant non-dimensional quantities, relating to meter resets and maxima, may now be defined. For the rth visible meter of the k-spot jackpot game, in which both secondary and auxiliary meters exist, define the quantity $V_x(k,r,0)$ by eq (7). $V_x(k,r,0)$ is the equivalent number of player contributions to achieve the starting pool $v_x(k,r,0)$ for jackpot $\#x$, that is, it is the non-dimensional reset value for (k,r) jackpot $\#x$. Note that the starting pool $v_x(k,r,0)$ is always at least $P_{\min}(k,r)$, but in general will also include a contribution from the previous (k,r) auxiliary meter. In any event, it never exceeds $P_{\max}(k,r)$.

We also define $V^{\max}(k,r)$, the *non-dimensional ceiling*, or the equivalent number of player contributions for the (j,k) jackpot meter to reach to $P_{\max}(k,r)$,*if it started at zero*, by eq (8); $A^{\max}(k,r)$ is defined similarly. It is also convenient to define $A_x(k,r,0)$, the non-dimensional starting value for the auxiliary meter (eq (9)). In each of these definitions, we assume that the quantities on the right-hand sides are such that $V_x(k,r,0)$, $V^{\max}(k,r)$ and $A_x(k,r,0)$ are integers.

$$V_x(k,r,0) = \frac{v_x(k,r,0)}{y(k)\lambda(k,r)} \tag{7}$$

$$V^{\max}(k,r) = \frac{P_{\max}(k,r)}{y(k)\lambda(k,r)} \tag{8}$$

$$A_x(k,r,0) = \frac{a_x(k,r,0)}{y(k)\lambda(k,r)} \tag{9}$$

$$A^{\max}(k,r) = \frac{\sigma P_{\max}(k,r)}{y(k)\lambda(k,r)} \tag{10}$$

From these equations, it may be noted that the quantities $V^{\max}(k,r)$ and $A^{\max}(k,r)$ are independent of x, the jackpot sequence index.

Another useful quantity is obtained as follows. Noting that $v_1(0) = P_{\min}$, and substituting $x = 1$ in eq (7), and adopting the convention that $t_0 = 0$, we obtain eq (11).

$$V_1(0) = \frac{P_{\min}}{\lambda y} \tag{11}$$

Note that all quantities relevant to the analysis conducted in the following section are summarized in Table 2.

7 Analysis

We consider the (k, r) jackpot game for fixed k and r. Recall the definitions of $v_x(k, r, t_x)$ and $a_x(k, r, t_x)$ as the amounts in the visible and auxiliary meters respectively after t_x player contributions, and after $x-1$ jackpots have been won (the xth jackpot game is currently being played, and jackpot $\#x$ will be the next to be won). Now suppose that jackpot $\#x$ is won after t_x player contributions for that jackpot. If $x > 1$, then the amount of the previous jackpot win was $v_{x-1}(k, r, t_{x-1})$, and $a_{x-1}(k, r, t_{x-1})$ goes into the next jackpot pool, along with $P_{\min}(k, r)$, provided always that this sum does not exceed $P_{\max}(k, r)$.
Note that for jackpot $\#x$, the visible meter does not necessarily start at $P_{\min}(k, r)$, but at $v_x(k, r, 0)$, and also, if t_x is such that $v_x(k, r, t_x)$ exactly equals $P_{\max}(k, r)$, then the total amount of player contributions at that time is given by eq (12).

$$\frac{P_{\max}(k, r) - v_x(k, r, 0)}{y(k)\lambda(k, r)} = V^{\max} - V_x(0) \equiv \Delta V_x \tag{12}$$

Thus, ΔV_x is the number of player contributions necessary to exactly fill the visible meter, for jackpot $\#x$, given its starting value of $v_x(k, r, 0)$, or in non-dimensional terms, $V_x(0)$. It turns out that the task of expressing subdomains for t_x in the recurrences to follow is made much simpler by making use of the quantity just defined, ΔV_x. It is also convenient to proceed in a similar fashion for the auxiliary meter. Thus, if $a_x(k, r, t_x)$ reaches $\sigma P_{\max}(k, r)$, then the total amount of player contributions at that time is given by the expression (13).

$$\frac{(\sigma + 1) P_{\max} - v_x(0) - a_x(0)}{y\lambda} = (\sigma + 1) V^{\max} - V_x(0) - A_x(0) \tag{13}$$

$$= \sigma V^{\max} + \Delta V_x - A_x(0) \tag{14}$$

$$= \Delta V_x + \Delta A_x \tag{15}$$

In the last equation, we have used

$$\Delta A_x = \sigma V^{\max} - A_x(0) \tag{16}$$

This quantity, ΔA_x, is the number of player contributions necessary to exactly fill the auxiliary meter, for jackpot $\#x$, given its starting value of $a_x(k, r, 0)$, or in non-dimensional terms, $A_x(0)$.
For compactness, and convenience of typesetting, in the displayed equations, the dependence of most quantities on (k, r) in the analysis to follow is suppressed. It will be found convenient to re-introduce it for the section in which revenue, profit and hold are calculated.

7.1 Recurrence for visible meter $v_x(t_x)$

The visible meter satisfies the recurrence of eq (17).

$$v_x(t_x) = \begin{cases} v_x(0) + t_x y\lambda & \text{if} \quad 0 \le t_x \le \Delta V_x \\ P_{\max} & \text{otherwise} \end{cases} \tag{17}$$

7.2 Recurrence for auxiliary meter $a_x(t_x)$

In order to develop an expression for the contents of the auxiliary meter, a_x, we must consider three possible cases.

1. If jackpot $\#x$ is won before the visible meter has filled, then a_x remains at its initial value, $a_x(0)$. This will be the case if and only if the number of contributions t_x does not exceed ΔV_x; that is, if $0 \leq t_x \leq \Delta V_x$.

2. If jackpot $\#x$ is won after both visible and auxiliary meters have filled, then $a_x(t_x) = \sigma P_{\max}$, by definition of both σ and $P_{\max}$. According to eq (15), this will be the case if and only if $t_x > \Delta V_x + \Delta A_x$.

3. Otherwise, a_x will assume some intermediate value, given by the expression $a_x(0) + (t_x - \Delta V_x)\ y\,\lambda$.

These three cases are summarized by eq (18).

$$a_x(t_x) = \begin{cases} a_x(0) & \text{if} \quad 0 \leq t_x \leq \Delta V_x \\ a_x(0) + (t_x - \Delta V_x)\ y\,\lambda & \text{if} \quad \Delta V_x < t_x \leq \Delta V_x + \Delta A_x \\ \sigma P_{\max} & \text{if} \quad \Delta V_x + \Delta A_x < t_x \end{cases} \tag{18}$$

7.3 Visible meter reset $v_x(0)$

The initial value of the visible meter is given by eq (19).

$$v_x(0) = \min(P_{\max}, P_{\min} + a_{x-1}(t_{x-1})) \tag{19}$$

7.4 Auxiliary meter reset $a_x(0)$

The initial value of the auxiliary meter, $a_x(0)$, is given by eq (20).

$$a_x(0) = \begin{cases} 0 & \text{if} \quad 0 \leq a_{x-1}(t_{x-1}) \leq P_{\max} - P_{\min} \\ a_{x-1}(t_{x-1}) - (P_{\max} - P_{\min}) & \text{if} \quad P_{\max} - P_{\min} < a_{x-1}(t_{x-1}) \end{cases} \tag{20}$$

For example, if $\sigma = 2$, and the auxiliary meter from jackpot game $x-1$ is full, i.e., $a_{x-1}(t_{x-1}) = 2P_{\max}$, then the largest amount that can go into the visible meter for jackpot game x is $P_{\max} - P_{\min}$, and the largest amount that can go into the auxiliary meter for jackpot game x is $P_{\max} + P_{\min}$. Note that this latter amount, $P_{\max} + P_{\min}$, is more than enough to fill the visible meter for game $x+1$, even if jackpot $\#x$ is won immediately.

7.5 Non-dimensional visible meter reset $V_x(0)$

Equation (19) is valid for $x \geq 2$, however, if we conveniently define $a_0(t_0) = 0$, then it becomes valid for $x \geq 1$. From equations (7) and (19), we obtain:

$$y\lambda V_x\,(0) = \min(P_{\max}, P_{\min} + a_{x-1}(t_{x-1})) \tag{21}$$

Applying the definition of $a_x(t)$ from eq (18) (replacing x by $x-1$ therein), and making use of eqs (6) and (8), we obtain the following (again omitting details of (k,r) dependence), for $x \geq 2$.

$$V_x\,(0) = \min\left(V^{\max}, V_1\,(0) + A_{x-1}\,(t_{x-1})\right) \tag{22}$$

It is possible to compute (see section 10) an expected value for $V_x\,(0)$, but this will be in terms of $V_{x-1}\,(0)$, which, in turn, will be in terms of t_{x-2}. Also, we see that, in general, the number of contributions to the jackpot pool before the jackpot is won depends on the value of the auxiliary meter, which, in turn, depends on the number of contributions to the jackpot pool in the previous jackpot game, which, in turn depends on, ... etc. Under some circumstances, such a chain of dependencies leads to recurrence relations which may be solved algebraically in closed form. In more intricate cases, such as the present one, we resort to a computer simulation. The relevant recurrences for the simulation will be derived in the paragraphs to follow.

7.6 Non-dimensional visible meter $V_x(t_x)$

Inspection of equations (8), (17), (5) leads us to a simple recurrence for the new non-dimensional quantity $V_x(t_x)$.

$$V_x(t_x) = \begin{cases} V_x(0) + t_x & \text{if} \quad 0 \leq t_x \leq \Delta V_x \\ V^{\max} & \text{otherwise} \end{cases} \tag{23}$$

7.7 Non-dimensional auxiliary meter $A_x(t_x)$

We have $A_1\,(t_1) = 0$, and for $x \geq 2$, from eqs (6), (12), (8), (18), we obtain eq (24).

$$A_x(t_x) = \begin{cases} A_x(0) & \text{if} \quad 0 \leq t_x \leq \Delta V_x \\ A_x(0) + t_x - \Delta V_x & \text{if} \quad \Delta V_x < t_x \leq \Delta V_x + \Delta A_x \\ \sigma V^{\max} & \text{if} \quad \Delta V_x + \Delta A_x < t_x \end{cases} \tag{24}$$

7.8 Non-dimensional auxiliary meter reset $A_x(0)$

We have $A_1\,(0) = 0$, and for $x \geq 2$, from eqs (20) and (8), we obtain eq (25).

$$A_x(0) = \begin{cases} 0 & \text{if} \quad 0 \leq A_{x-1}(t_{x-1}) \leq \Delta V_1 \\ A_{x-1}(t_{x-1}) - \Delta V_1 & \text{if} \quad \Delta V_1 < A_{x-1}(t_{x-1}) \end{cases} \tag{25}$$

7.9 Revenue, profit, and hold calculation

It is convenient in this section to reintroduce explicit dependence on (k, r). We now consider the profit, $\pi_x(k, r)$, on jackpot $\#x$, which is given by the following expression.

$$\pi_x(k, r) = \lambda(k, r)\, y(k) t_x \;-\; v_x(k, r, t_x) \tag{26}$$

Given the requisite parameters and a value for t_x, a hold could easily be calculated for jackpot $\#x$, however, we suppose it is of more interest to calculate holds after a long series of jackpot simulations. Therefore, suppose that X jackpots in total have been won. Then the total accumulated payout is given by the expression (27).

$$\sum_{x=1}^{X} v_x(k, r, t_x) \tag{27}$$

Total revenue from these X jackpots is given by the expression (28).

$$\sum_{x=1}^{X} \lambda(k, r)\, y(k) t_x \tag{28}$$

Net profit from these X consecutive jackpots is therefore given by the summed difference (29).

$$\sum_{x=1}^{X} \left(\lambda(k, r)\, y(k) t_x - v_x(k, r, t_x)\right) \tag{29}$$

Thus, we see that, after X jackpots of type (k, r) have been won, the cumulative (long-term) hold, $H_X(k, r)$, is given by eq (30).

$$H_X(k, r) = \frac{\text{net profit after } X \text{ jackpots}}{\text{total revenue after } X \text{ jackpots}} \times 100\% \tag{30}$$

By employing eqs (28) and (29), this may be expressed as follows.

$$H_X(k, r) = \frac{\sum_{x=1}^{X} \pi_x(k, r)}{\sum_{x=1}^{X} \lambda(k, r)\, y(k) t_x}$$

This reduces to:

$$H_X(k, r) = 1 - \frac{\sum_{x=1}^{X} v_x(k, r, t_x)}{\sum_{x=1}^{X} \lambda(k, r)\, y(k) t_x} \tag{32}$$

In terms of the *non-dimensional visible meter*, $V_x(k, r, t_x)$ (see eq (5)), the expression in eq (32) for the hold may be written as follows.

$$H_X(k, r) = 1 - \frac{\sum_{x=1}^{X} V_x(k, r, t_x)}{\sum_{x=1}^{X} t_x} \tag{33}$$

For a given (k, r), the recurrence (23) may be used in the simulation to compute the sequence of values $V_x(t_x)$ as x runs from 1 to its maximum value X. The hold for X jackpots may then be computed from eq (33).

7.10 Overall hold for bet-type k

For the overall hold $H_X(k)$ for all possible catches in the $k-spot$ game, we need to re-introduce the explicit dependence of t_x on the pair (k, r), i.e. $t_x = t_x(k, r)$. We then compute:

$$H_X(k) = \frac{\sum_{r=0}^{k} \sum_{x=1}^{X} \pi_x(k, r)}{\sum_{r=0}^{k} \sum_{x=1}^{X} \lambda(k, r)\, y(k) t_x(k, r)} \tag{34}$$

This is just total profit divided by total receipts (revenue), each taken over all catches for the fixed bet-type, k. It can be further shown that eq (34) is equivalent to:

$$H_X(k) = 1 - \frac{\sum_{r=0}^{k} \lambda(k, r) \sum_{x=1}^{X} V_x(k, r, t_x(k, r))}{\sum_{r=0}^{k} \lambda(k, r) \sum_{x=1}^{X} t_x(k, r)} \tag{35}$$

In the simulator program (described below), the hold for each r, for the current bet-type, k, is calculated as $H_X(k, r)$ from eq (33). To find the net hold across all catches for bet-type k, we use eq (35).

8 Definition of symbols

The major variables used in the simulation model are summarized in Table 2.

9 The simulator program

Since the probabilistic recurrences detailed in the foregoing sections are very awkward to solve for the long-term hold percentage in closed form, it was decided to create a computer simulation. This was developed very effectively in

Table 2: Definition of symbols

Name	**Meaning**
K	set of bet types, $K = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 15, 20, 40\}$
K'	set of jackpot bet types; presently $K' = \{7, 8, 9, 10\}$
k	# spots (bet-type); $k \in K'$ for jackpot game
r	# player hits (catch); $0 \leq r \leq k$
x	jackpot counter
$t_x(k, r)$	# player contributions when (k, r) jackpot #x is won
$\lambda(k, r)$	fractional contribution to (k, r) jackpot meter
$P_{\min}(k, r)$	minimum payout for catch r in (k, r) jackpot game
$P_{\max}(k, r)$	maximum payout for catch r in (k, r) jackpot game
$v_x(k, r, t_x)$	visible meter r when (k, r) jackpot # x is won
$a_x(k, r, t_x)$	auxiliary meter r when (k, r) jackpot # x is won
$V^{\max}(k, r)$	non-dimensional maximum payout in (k, r) jackpot game
$V_x(k, r, t_x)$	non-dimensional visible meter
$A_x(k, r, t_x)$	non-dimensional auxiliary meter
$H_X(k, r)$	long-term hold for catch r of k-spot jackpot game
$H_X(k)$	overall long-term hold for k-spot jackpot game
$\pi_x(k, r)$	profit for (k, r) jackpot # x
$y(k)$	unit player contribution to k-spot jackpot game
σ	ratio of max auxiliary to max visible meter (all bet types)
ΔV_x	# player contributions to take visible meter from reset to full
ΔA_x	# player contributions to take auxiliary meter from reset to full

Borland's Delphi-32 object-oriented programming environment. Two key features of this system are easy creation of user-interface elements, and, importantly for a computationally-intensive application such the present one, efficient executable code.

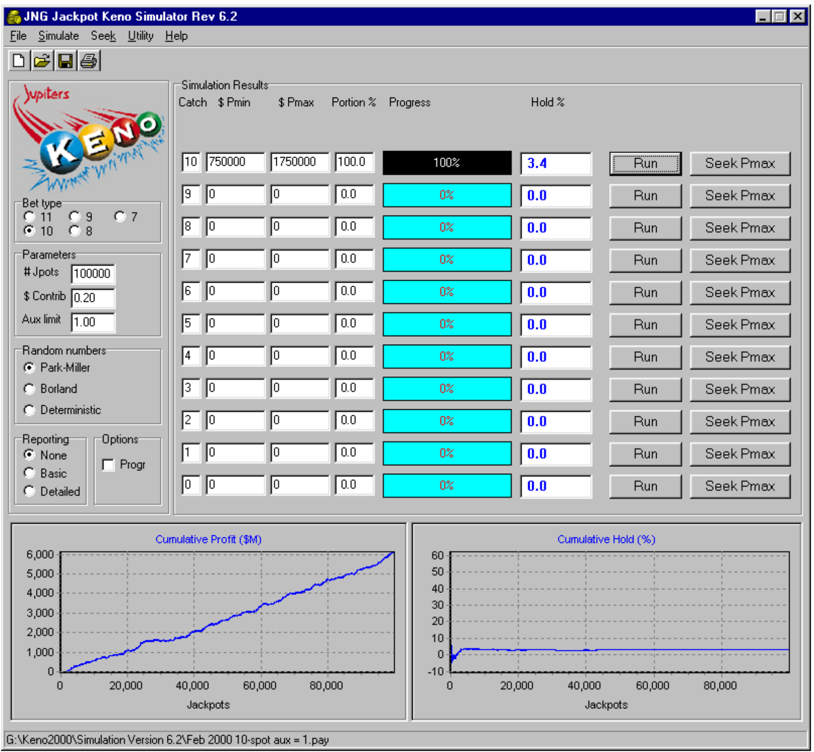

Figure 1: Jackpot KENO Simulator User Interface

In the simulator program, (see Figure 1), it is possible to run an individual (k, r) simulation, and also to invoke simulation of all possible catches r, for a given bet-type, k. For the case of an individual (k, r) simulation, it is convenient to remove as much of the dependence of simulation quantities on $\lambda(k, r), y(k), P_{\min}(k, r), P_{\max}(k, r)$ as possible, so that most quantities of interest are expressed in non-dimensional form. This approach has at least two positive implications. Firstly, it allows for somewhat simplified equations and recurrences, based on those already derived in the foregoing paragraphs. Secondly, the simulator algorithm and subsequent code are both simplified.

The simulator also allows both for determination of long-term hold given payscale, and the inverse: determination of payscale given long-term hold; this latter mode of operation is, naturally, of great interest.

10 Generation of t_x in the simulator program

For the computer simulation of a sequence of jackpot wins to proceed, it is necessary to have access to a random number generator which will give a sequence of values for t_1, t_2, $t_3, ...$ chosen as follows. Within each jackpot, the number of contributions, s, before a jackpot is won has a geometric distribution. Then, for $s \in \{1, 2, 3, \ldots\}$

$$g(s; p) = p(1-p)^{s-1} = pq^{s-1}$$

is the probability that s tickets will be sold (contributions made) before the jackpot is won. For the xth jackpot, $t_x$thus represents the number of contributions before it is won. Generation of simulated values of t_x is done in the following way: firstly, a uniform random number generator provides a sequence of numbers $\alpha_1, \alpha_2, \alpha_3, ...$, each in the interval $0 \leq \alpha_x \leq 1$. Then, since $\sum_{s=1}^{t_x} pq^{s-1}$ gives the probability of at most t_x trials between wins, for each such α_x we then compute the smallest integer, t_x, such that:

$$\sum_{s=1}^{t_x} pq^{s-1} \geq \alpha_x \tag{36}$$

i.e.

$$\frac{p(1-q^{t_x})}{1-q} \geq \alpha_x \tag{37}$$

i.e.

$$1 - q^{t_x} \geq \alpha_x \tag{38}$$

We obtain:

$$t_x = \left\lceil \frac{\ln(1-\alpha_x)}{\ln q} \right\rceil \tag{39}$$

The sequence of values, $\{t_x\}$, generated in this fashion provides the simulator program with the number of player contributions before (k, r) jackpot $\#x$ is won, for $1 \leq x \leq X$. For the pseudo-random uniform generator, the simulator program is able to make use of either the Park-Miller uniform generator [8], which is implemented in Delphi, or Delphi's intrinsic uniform generator. Virtually indistinguishable results are obtained for these two generators.

Remark 1 *For the game in which the jackpot is won, we assume that the unsuccessful tickets (selections of numbers) precede the successful one(s) in the sequence of Bernoulli trials. Otherwise, strictly, the contributions from these tickets would have to go into the meter for the next jackpot, whereas we have counted them all as going into the current jackpot. In other words, unless we make this assumption, we would be very slightly overestimating the jackpot for the current game, and equally underestimating the pool for the next one. In practice it will make no effective difference.*

Table 3: $\sigma = 1$

Bet-type	7-spot	8-spot	9-spot	10-spot
Pmin	$ 1,000	$ 2,000	$ 35,000	$ 750,000
Pmax	$ 7,000	$ 30,000	$ 115,000	$ 1,750,000
Hold for simulation #1	14.9%	35.1%	58.4%	1.9%
Hold for simulation #2	14.6%	34.7%	58.4%	1.6%
Hold for simulation #3	14.4%	35.3%	58.4%	1.8%
Hold for simulation #4	14.4%	35.0%	58.4%	2.0%
Hold for simulation #5	14.6%	35.0%	58.5%	1.4%
Hold for simulation #6	14.2%	34.2%	58.4%	1.7%
Hold for simulation #7	14.8%	35.0%	58.4%	1.1%
Hold for simulation #8	14.4%	34.6%	58.3%	2.0%
Hold for simulation #9	15.0%	34.9%	58.5%	2.3%
Hold for simulation #10	14.7%	35.0%	58.2%	2.1%
Average jpot hold	14.60%	34.88%	58.39%	1.79%

11 Algorithm for inverse problem

A stochastic binary search technique is used to solve the inverse problem of determining payscale given hold. More precisely, the value of $P_{\max}$ is sought, as this has a much greater influence on the hold than does $P_{\min}$. As noted, the algorithm is coded in Borland's Delphi and converges to within 0.1% in all cases of interest to date. It may be compared to the well-known "shooting methods" used in the solution of ordinary differential equations, but of course, has a stochastic component, and if tolerance is reduced too much, convergence may fail. Indeed, even if convergence is achieved, there is no guarantee that such a solution is reproducible. It is therefore recommended that the technique be used to give a "ballpark estimate" or starting point for the payscale, and then the conventional forward simulation be used to refine it.

12 Experimental results

Results of simulations are shown in Tables 3 and 4. All jackpot holds here were obtained with simulation using 100,000 iterations, using Park-Miller random-number generator [8].

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Table 4: $\sigma = 2$

Bet-type	7-spot	8-spot	9-spot	10-spot
Pmin	$ 1,000	$ 2,000	$ 35,000	$ 750,000
Pmax	$ 7,000	$ 30,000	$ 115,000	$ 1,750,000
Hold for simulation #1	14.6%	35.3%	58.6%	1.7%
Hold for simulation #2	14.9%	34.8%	58.3%	2.0%
Hold for simulation #3	14.5%	34.7%	58.3%	1.4%
Hold for simulation #4	14.7%	34.9%	58.3%	1.4%
Hold for simulation #5	14.5%	35.1%	58.4%	1.8%
Hold for simulation #6	14.6%	34.8%	58.1%	1.7%
Hold for simulation #7	14.6%	34.8%	58.4%	2.6%
Hold for simulation #8	14.9%	34.6%	58.3%	2.5%
Hold for simulation #9	14.4%	35.2%	58.3%	2.0%
Hold for simulation #10	15.0%	34.9%	58.4%	2.0%
Average jpot hold	14.67%	34.91%	58.34%	1.91%

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