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Abstract

In a carrot game for a player, that player must help his opponent to get a desired response. In a stick game for a player, that player must hurt his opponent to get a desired response. We show that most all smooth monotonic two player games can be classi¯ed as: carrot for both, carrot for one and stick for the other, or stick for both. We transform carrot and stick games into sequential move games and incentive games. A carrot game for a player implies mutual bene¯ts will come from his moving ¯rst. A stick game for a player implies that if he moves ¯rst, the other player will su®er. Incentive games based on a carrot game for each player bring mutual gains. Those based on a stick game for each player bring harm to at least one and possibly both.

1   Introduction

In the Industrial Organization literature there is often reference to the di®erence between strategic substitutes and strategic complements (See Bulow, et. al.(1985), or Shy (1995)). A game is said to involve strategic substitutes if the best response functions are downward sloping. Alternatively, a game is said to involves strategic complements if the best response functions are upward sloping. Quantity competition games typically involve strategic substitutes and price competition games typically involve strategic complements.

One well know result is that under quantity competition, sequential movement bene¯ts the leader at the expense of the follower (Dowrick (1985) or Gal-Or (1985)). For price competition sequential movement bene¯ts both ¯rms and typically the follower bene¯ts more. While one can describe this di®erence in terms of strategic complements and strategic substitutes, this description has virtually no economic content. Furthermore, to come to the conclusion that strategic complements imply sequential movement will always be mutually bene¯cial would be misleading. We give an example of an advertising game with strategic complements where sequential movement never bene¯ts both ¯rms (Example 3.3).

In this paper I introduce the notions of carrots and sticks to describe differences between games like quantity and price competition games. Loosely speaking, a carrot game for a player is one in which he must help out his opponent in order to get a desired response. A stick game for a player, on the

other hand, is one in which he must harm his opponent in order to get a desired response. These definitions are inspired by the age old question of whether to use a carrot or a stick to motivate a beast of burden.

Once we recognize that price competition is a carrot game and quantity competition is a stick game, it is a trivial result that sequential movement will be mutually beneficial under price competition while only unilaterally beneficial under quantity competition (Proposition 2.1). In a carrot game the leader offers a carrot to his opponent to get a desired response and both benefit. Under price competition the carrot comes in the form of a higher price which translates into reduced competition. In a stick game the leader will whack his opponent with a stick to get a desired response. The stick comes in the form of a greater output level by the leader in a quantity setting game. This drives the market price and profit of the follower down.

Price and quantity competition games are typically modelled as smooth monotonic games. In Section 3 we define smooth monotonic games formally and show that most all of them can be classified in terms of carrots and sticks. We also give conditions under which games with strategic complements are carrot games, and games with strategic substitutes are stick games. In those cases, the distinction between strategic complements and strategic substitutes has a clear economic meaning which derives from the distinction between carrot and stick games.

In our analysis of carrot and stick games we allow the players to be asymmetric. In fact, a game may be carrot for one player and stick for the other. Such an example is a duopoly game where one firm chooses a price and the other chooses a quantity (Example 3.2). It turns out that this game is a carrot game for the quantity setter and a stick game for the price setter. Whether or not sequential movement is mutually beneficial here depends on who moves first. The game will be mutually beneficial only in the case that the quantity setter moves first.

In relation to the literature on sequential versus simultaneous movement, Hamilton and Slutsky (1990) endogenized the choice between simultaneous and sequential movement. We do not explicitly give players the choice between simultaneous or sequential movement, but clearly our results have implications for that choice. Our results are given conditional on whether a game has a carrot or a stick nature while Hamilton and Slutsky's results are given conditional on whether sequential movement is mutually or unilaterally beneficial. As is shown in Proposition 2.1, the carrot or stick nature of a game determines whether or not sequential movement will be mutually beneficial.

Furthermore, as we will see in Section 4, carrot and stick games have implications about other types of transformations of games, not just from simultaneous moves to sequential moves.

For example, many results in the industrial organization literature about manipulating payoffs in games are easily understood as coming from the carrot or stick nature of the original game. Examples include owners of firms who pay managers to act more or less aggressively than profit maximizers (See Fershtman and Judd (1987), Sklivas (1987), Vickers (1984), Basu (1995), and Basu et. al. (1997)). Additionally, Bonanno and Vickers (1988), and Rey and Stiglitz

(1995) show how vertical separation may provide firms with new commitment opportunities that yield higher equilibrium profits. Baye, Crocker and Ju (1996) show similar results for horizontal separation.

The main results of the papers cited in the previous paragraph can be predicted immediately from noting the carrot or stick nature of the original games. Take the example of the managerial incentives game studied by Sklivas (1987). The original game is a carrot game for each firm when the firms compete in prices and a stick game for each firm when they compete in quantities.

We show in a general setting that when players can manipulate incentives of their agents in an incentive game, they will choose to magnify the carrot or stick behavior of their agents (Proposition 4.1). Thus, if the original game is a stick game for one firm (player), then the manager (agent) of that firm will be given incentive to act more aggressively (more stick like) in the incentive game. Alternatively, if the original game is a carrot game for a firm, then the manager of that firm will be given incentive to act less aggressively (more carrot like) in the incentive game.

If both firms compete in prices, it follows from the carrot nature of the original game that an incentive game will lead to even more carrot like behavior and greater firm profits (Proposition 4.2). Alternatively, if the firms compete in quantities, the incentive game will lead to even more stick like behavior and reduce the profits of at least one firm (Proposition 4.3). Note the distinction between the result for carrot games and the result for stick games. If the original game is a carrot game for each player, then the incentive game will raise the welfare of each. However, if the original game is a stick game for each player, then the welfare of one, but possibly not both will be reduced.

For example, we find that if the managers compete in quantities, but one has a cost advantage over the other, then the firm with the cost advantage can actually do better in the incentive game (Example 4.2). The result that both firms are made worse off from the incentive game based on a stick game for each player is obtained if the players are symmetric (Corollary 4.2).

The layout of the paper is as follows. In Section 2 we define carrot and stick games and discuss the implications of these games for sequential movements. In Section 3 we concentrate on smooth monotonic games and relate our results for those games to strategic complements and strategic substitutes. In Section 4 we give some welfare results for incentive games based on smooth monotonic games. In Section 5 we show the relationship between carrot and stick games and Stackelberg solvable games. Conclusions are given in Section 6. All proofs are in the appendix.

2   Carrots and Sticks

We focus on a two person normal form game $G = ((S_1; S_2); (u_1; u_2))$ where the subscripts 1 and 2 denote the players. The strategy set of player i for $i = 1; 2$ is S_i and we refer to a specific strategy of player i as s_i. The objective of each player is to maximize his payoff $u_i : S_1 \times S_2 \rightarrow R$, where R is the real line. Let $b_1(s_2) = \{s_1 \mid s_1 \in S_1 \text{ and } 8 s_1' \in S_1; u_1(s_1; s_2) \geq u_1(s_1'; s_2)\}$. We call $b_1(s_2)$

the best response set of player 1 given s_2. We assume that for each s_2 the best response set is a singleton and thus we can use $b_1(s_2)$ to denote the best response function of player 1 and $b_2(s_1)$ to denote the best response function of player 2. A Nash equilibrium in a game G is a pair of strategies s_1^π and s_2^π that simultaneously satisfy: i) $u_1(s_1^\pi; s_2^\pi) \geq u_1(s_1; s_2^\pi)$ for all $s_1 \in S_1$, and ii) $u_2(s_1^\pi; s_2^\pi) \geq u_2(s_1^\pi; s_2)$ for all $s_2 \in S_2$:

We de¯ne $\underline{v}_1$ to be the minimum Nash equilibrium payo® of player 1 in G and ∇_1 to be the maximum Nash equilibrium payo® of player 1 in G. We de¯ne $\underline{v}_2$ and ∇_2 in the same way for player 2 and assume that these values exist for both players. The game of Figure 2.1 has a unique equilibrium $(T; L)$, so $\underline{v}_1 = \nabla_1 = 2$, and $\underline{v}_2 = \nabla_2 = 3$. In general, each of $\underline{v}_1$, ∇_1, $\underline{v}_2$, and ∇_2 may be determined by a di®erent Nash equilibrium.

<table>
<tr><td></td><td colspan="3">Figure 2.1
Player 2</td></tr>
<tr><td></td><td></td><td>L</td><td>R</td></tr>
<tr><td rowspan="2">Player 1</td><td>T</td><td>2,3</td><td>6,1</td></tr>
<tr><td>B</td><td>1,-3</td><td>3,2</td></tr>
</table>

<table>
<tr><td></td><td colspan="3">Figure 2.2
Player 2</td></tr>
<tr><td></td><td></td><td>L</td><td>R</td></tr>
<tr><td rowspan="2">Player 1</td><td>T</td><td>1,1</td><td>3,-1</td></tr>
<tr><td>B</td><td>0,0</td><td>2,2</td></tr>
</table>

We now de¯ne carrot and stick games for player 1. Carrot and stick games for player 2 are de¯ned in an analogous way by exchanging the roles of the players. In the remainder of the paper we will often give results only for player 1.

(2.1) **Carrot Game:** A carrot game for player 1 is one in which $u_1(s_1; b_2(s_1)) > \nabla_1$ implies $u_2(s_1; b_2(s_1)) > \nabla_2$, and there exists at least one $s_1 \in S_1$ for which $u_1(s_1; b_2(s_1)) > \nabla_1$.

(2.2) **Stick Game:** A stick game for player 1 is one in which $u_1(s_1; b_2(s_1)) > \nabla_1$ implies $u_2(s_1; b_2(s_1)) < \underline{v}_2$, and there exists at least one $s_1 \in S_1$ for which $u_1(s_1; b_2(s_1)) > \nabla_1$.

Loosely speaking a carrot game for player 1 is one in which he must help player 2 in order to get a desired response. A stick game for player 1 is one in which he must hurt player 2 in order to get a desired response.

While every two player game cannot be classi¯ed as a stick or carrot game for each player, many games of interest to economists and game theorists can. For example, in Section 3 we show that a heterogeneous price setting game with linear demand and constant marginal cost is an example of a carrot game for each player while the quantity setting version is a stick game for each player (Example 3.1).

Let's ¯rst look at some simple games. The game of Figure 2.1 turns out to be a stick game for player 1 but neither carrot nor stick for player 2. Let's check. Recall from our previous argument that the unique Nash equilibrium $(T; L)$ determines $\underline{v}_1 = \nabla_1 = 2$, and $\underline{v}_2 = \nabla_2 = 3$. Clearly, since $b_2(T) = L$ and $b_2(B) = R$, the strategy $s_1 = B$ is the only strategy of player 1 that satis¯es $u_1(s_1; b_2(s_1)) > \nabla_1$. Since $u_2(B; b_2(B)) = 2 < \underline{v}_2$, it follows that this is a stick

game for player 1. For player 2, the game is neither stick nor carrot since there does not exist a strategy $s_2 \in \{L, R\} = S_2$ for which $u_2(b_1(s_2), s_2) > \bar{v}_2$.

By a similar type of analysis we find that the game of Figure 2.2 is a carrot game for player 1 and neither carrot nor stick for player 2.

What will happen if player 1 is allowed to move first? In the game of Figure 2.1, he will choose B rather than T, inducing the desired response of R by player 2. Player 1's payoff will rise from 2 to 3 while player 2's payoff falls from 3 to 2.

In the game of Figure 2.2, if player 1 can move first he will again choose B inducing a desired response of R. Here both players benefit from this change since the payoff to each increases from 1 to 2.

In general, one player's gain from sequential movement may or may not be at the expense of another player. Note that the player moving first must do no worse from this change since he can simply choose the strategy that yields his value in the original game G.

Given a game $G = ((S_1, S_2), (u_1, u_2))$, a sequential move game $G^0 = ((S_1, F_2), (u_1, u_2))$ is obtained by allowing player 1 to move first and player 2 to observe 1's strategy choice before he makes a choice. In the game G^0 a strategy of player 2, which we denote by f_2, describes what player 2 will do for each strategy s_1 of player 1. The strategy set of player 2 is thus a set of functions $F_2 = \{f_2 \mid f_2 : S_1 \to S_2\}$. This captures the idea that player 2 can observe and react to player 1's choice. The appropriate equilibrium concept in this setting is a subgame perfect equilibrium.

Since we have assumed the best response sets are unique valued, a strategy profile (s_1^*, f_2^*) is a subgame perfect equilibrium in G^0 if and only if $f_2^*(s_1) = b_2(s_1)$ and $u_1(s_1^*, b_2(s_1^*)) \geq u_1(s_1, b_2(s_1))$ for all $s_1 \in S_1$. Another sequential move game can be obtained by allowing player 2 to move first.

Carrot games, on the one hand, are games where both players gain from sequential movement, while stick games involve one gaining at the other's expense. This idea is expressed in the next proposition.

Proposition 2.1. Let G be a two person game and let G^0 be the sequential move game where player 1 moves first.

(2.3) If G is a carrot game for player 1, then for any subgame perfect equilibrium (s_1^*, f_2^*) of G^0, $u_1(s_1^*, f_2^*(s_1^*)) > \bar{v}_1$ and $u_2(s_1^*, f_2^*(s_1^*)) > \bar{v}_2$.

(2.4) If G is a stick game for player 1, then for any subgame perfect equilibrium (s_1^*, f_2^*) of G^0, $u_1(s_1^*, f_2^*(s_1^*)) > \bar{v}_1$ and $u_2(s_1^*, f_2^*(s_1^*)) < \underline{v}_2$.

If we can establish that a game is a carrot or stick game for a player we will be able to tell immediately from Proposition 2.1 whether or not sequential movement with that player moving first will be unilaterally or mutually beneficial. Notice that the result states that sequential movement in a carrot game will improve on each player's best Nash equilibrium payoff in G. In a stick game sequential movement makes the player moving first better off than his best Nash equilibrium payoff in G while it makes the player moving second worse off than his worst Nash equilibrium payoff in G.

You may have noticed that in each of the simple 2x2 examples given above, the game was neither a carrot nor a stick game for player 2. For 2x2 games, it turns out that we can classify the game in terms of carrots and sticks for at most one of the two players.

Lemma 2.1: Let G be a 2x2 game. If ∇_1 and ∇_2 are defined by the same Nash equilibrium, then G is a carrot or stick game for at most one player. Furthermore, if ∇_1 and ∇_2 are defined by different Nash equilibria, then G is neither a carrot nor a stick game for either player.

In view of this result, it appears there may be a problem classifying games with finite strategy spaces as carrot or stick games for both players. This problem is really only one of 2x2 games. The next example, which is based on the game of Figure 2.1, is meant to dispel the belief that such a result holds more generally.

Figure 2.3

Player 2

		L	R	N
Player 1	T	2,3	6,1	0,0
	B	1-3	3,2	0,0
	N	0,5	0,0	α,4

The game of Figure 2.3 is a stick game for player 1 for all $\alpha > 0$. However, for player 2 it is a stick game if $0 < \alpha < 2$ and a carrot game if $\alpha > 2$.

In the case when $\alpha > 2$, since the game is a carrot game for player 2 and a stick game for player 1, Proposition 2.1 informs us that sequential movement will only be mutually beneficial if player 2 moves first. We will discuss other asymmetric examples in the next section.

3   Smooth Monotonic Games

We now turn our attention to smooth monotonic games. These games are quite commonly used throughout the industrial organization and economics literature. We show that all smooth monotonic games where each player has something to gain from moving first, e.g., those with only interior equilibria, can be classified as either a carrot or a stick game for each player.

Let the strategy space of each player be some convex subset of the real line R. Let the payoff functions $u_1(s_1; s_2)$ and $u_2(s_1; s_2)$ and the best response functions $b_1(s_2)$ and $b_2(s_1)$ be differentiable. We use $b_1'(s_2)$, and $b_2'(s_1)$ to represent the first derivatives of the best response functions.

Smooth Monotonic Game: A two player game G is a smooth monotonic game iff the following are true at all $(s_1; s_2)$:

(3.1) Smoothness: the payoff functions $u_1(s_1; s_2)$, $u_2(s_1; s_2)$, and the best response functions $b_1(s_2)$ and $b_2(s_1)$ are differentiable.

(3.2) **Monotonic Externalities:** i) either $\frac{@u_1(s_1;s_2)}{@s_2} < 0$ everywhere or $\frac{@u_1(s_1;s_2)}{@s_2} > 0$ everywhere; and ii) either $\frac{@u_2(s_1;s_2)}{@s_1} < 0$ everywhere or $\frac{@u_2(s_1;s_2)}{@s_1} > 0$ everywhere.

(3.3) **Monotonic Best Responses:** i) either $b_1^0(s_2) < 0$ everywhere or $b_1^0(s_2) > 0$ everywhere; and ii) either $b_2^0(s_1) < 0$ everywhere or $b_2^0(s_1) > 0$ everywhere.

When it happens that $\frac{@u_1(s_1;s_2)}{@s_2} < 0$, we interpret player 2 as imposing a negative externality on player 1. When the sign is positive, player 2 imposes a positive externality on player 1. The following is an example of a smooth monotonic game.

Example 3.1: (Price or Quantity competition): Two ¯rms sell heterogeneous substitute products, face linear demand curves and constant zero marginal cost. For quantity competition the inverse demand system is given by: $p_i(q_i; q_j) = 1 ¡ q_i ¡ \frac{q_j}{2}$, for $i = 1; 2$ and $j ø i$. For price competition the demand system is given by: $q_i(p_i; p_j) = \frac{2}{3} ¡ \frac{4p_i}{3} + \frac{2p_j}{3}$, for $i = 1; 2$ and $j ø i$.[1] The strategy space in either case is the strictly positive real line R^{++}.

It is straightforward to check that the games are smooth and monotonic. In the case of quantity setting, the externalities are negative since raising one player's quantity lowers the market price faced by the other ¯rm. In the price setting game, the externalities are positive since raising your price is a less competitive act. The best response functions, which are given in Figures 3.1 and 3.2, are also clearly monotonic.

Figure 3.1: Quantity Competition

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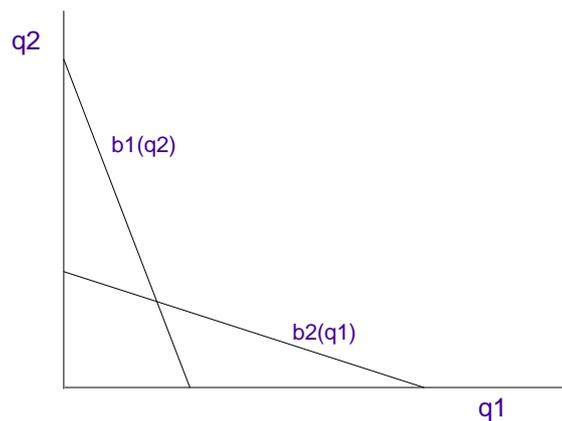
[1]These come from the more general linear demand system given by Klemperer and Meyer (1986): $p_i = ® ¡ ¯q_i ¡ °q_j$, for $i = 1; 2; j ø i$, when we let $® = ¯ = 1$, and $° = \frac{1}{2}$.
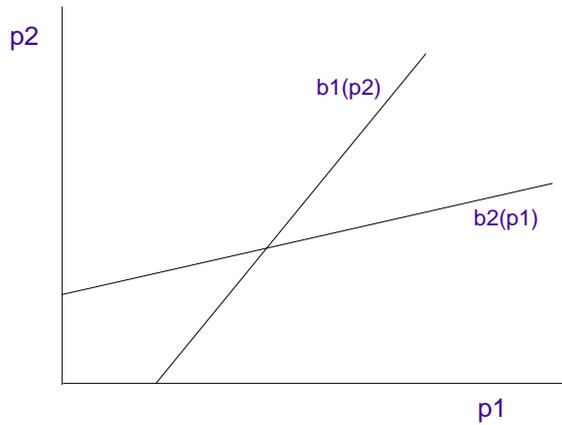
In the price and quantity games of Example 3.1 the players are very symmetric. In general the players may be quite asymmetric. We give two situations involving asymmetric players in Examples 3.2 and 3.3.

Example 3.2 : (Mixed Price/quantity competition): The demand and cost conditions and the strategy spaces are the same as in Example 3.1 except here ¯rm 1's strategy is a quantity and Firm 2's strategy is a price. The demand system is now given by: $p_1(q_1; p_2) = \frac{1}{2} + \frac{p_2}{2} - \frac{3q_1}{4}$, and $q_2(q_1; p_2) = 1 - \frac{q_1}{2} - p_2$.

The best response functions are: $b_1(p_2) = \frac{1}{3} + \frac{p_2}{3}$, and $b_2(q_1) = \frac{1}{2} - \frac{q_1}{4}$. The best response functions are monotonic but one is negatively sloped and the other is positively sloped. These are given in Figure 3.3. The externalities are also monotonic but they are in opposite directions since just as in Example 3.1, price setters generate positive externalities and quantity setters generate negative externalities.

Figure 3.3: Price/Quantity Competition

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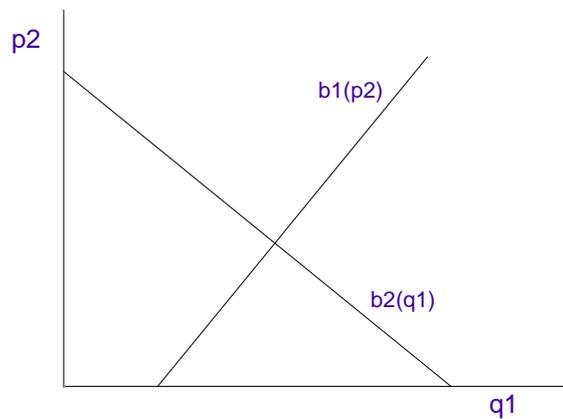
Example 3.3: (Advertising) Two stores can each choose a level of advertising. Larry (player 1) is located inside a shopping mall and can only be reached by first passing a larger store called Grand (player 2). Larry's advertising level a_1 is a positive externality ($\frac{\partial u_2}{\partial a_1} > 0$) for Grand since more people will stop at Grand on the way to Larry's. Grand's advertising level a_2 imposes a negative externality ($\frac{\partial u_1}{\partial a_2} < 0$) on Larry since Grand will steal customers from Larry.

More specifically, let Larry's utility function be given by $u_1(a_1; a_2) = b - \frac{a_2}{a_1} - a_1$, and let Grand's utility function be given by $u_2(a_1; a_2) = a_1 a_2 - \frac{a_2^2}{2} + 2a_2$. The strategy space of each player is again the strictly positive real line. Then it follows that the externalities move in opposite directions since $\frac{\partial u_1}{\partial a_2} = -\frac{1}{a_1} < 0$ and $\frac{\partial u_2}{\partial a_1} = a_2 > 0$. The best response functions shown in Figure 3.4 are both positively sloped since $b_1(a_2) = a_2^{1=2}$ and $b_2(a_1) = a_1 + 2$.

Figure 3.4: Advertising
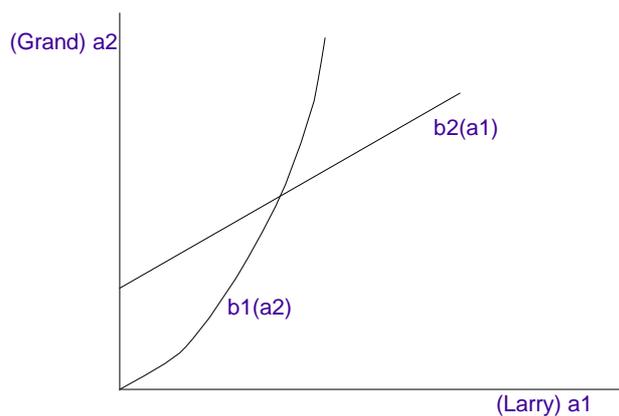
Proposition 3.1 is the main result of this section. It states that all smooth monotonic games where each player has something to gain from moving first can be classified for each player as a carrot or stick game. Following our policy from Section 2 we give the result for player 1 only.

Proposition 3.1. Let G be a smooth monotonic game with $u_1(s_1; b_2(s_1)) > \nabla_1$ for some s_1.

(3.4) If $\text{sign}\left[\frac{\partial u_1}{\partial s_2} b_2'(s_1)\right] = \text{sign}\left[\frac{\partial u_2}{\partial s_1}\right]$ then G is a carrot game for the player 1.

(3.5) If $\text{sign}\left[\frac{\partial u_1}{\partial s_2} b_2'(s_1)\right] \neq \text{sign}\left[\frac{\partial u_2}{\partial s_1}\right]$ then G is a stick game for player 1.

Proposition 3.1 gives us a simple way to determine whether a smooth monotonic game is a stick or a carrot game for a player when that player has

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something to gain from moving first. We simply compare $\text{sign}\left[\frac{\partial u_1}{\partial s_2} b_2'(s_1)\right]$ to $\text{sign}\left[\frac{\partial u_2}{\partial s_1}\right]$.

For smooth monotonic games, interior solutions suffice to ensure that each player has something to gain from moving first. To see why this is true consider the case when we are at an interior Nash equilibrium (s_1^π, s_2^π). If player 1 were to alter his strategy choice marginally and player 2 were to respond optimally to this change, then the change in player 1's utility would be given by $\frac{\partial u_1(s_1^\pi, s_2^\pi)}{\partial s_1} + \frac{\partial u_1(s_1^\pi, s_2^\pi)}{\partial s_2} b_2'(s_1^\pi)$. Since (s_1^π, s_2^π) is a Nash equilibrium it follows that the first term $\frac{\partial u_1(s_1^\pi, s_2^\pi)}{\partial s_1} = 0$. Since externalities are non-zero and best response functions have non-zero slope in smooth monotonic games, it follows that the second term $\frac{\partial u_1(s_1^\pi, s_2^\pi)}{\partial s_2} b_2'(s_1^\pi) \neq 0$. Hence, player 1 has something to gain by moving first. For example, if $\frac{\partial u_1(s_1^\pi, s_2^\pi)}{\partial s_2} b_2'(s_1^\pi) < 0$ then player 1 has something to gain by lowering his strategy marginally.

An example of when non-interior equilibria prohibit us from classifying smooth monotonic games in terms of carrot and sticks is a zero sum game. It is a well known result that each Nash equilibrium of a two person zero sum game involves each player minimizing his opponent's utility given his opponent's strategy. Monotonic externalities in a zero sum game imply that this minimization occurs when each player chooses an endpoint of his strategy space. We show in Section 5, that every zero sum two person game is not a carrot or a stick game for either player.

The simplicity of the comparison given in Proposition 3.1 can be understood on an intuitive level by again considering small changes around an interior Nash equilibrium point (s_1^π, s_2^π). From our earlier argument we know on the one hand that $\frac{\partial u_1}{\partial s_2} b_2'(s_1^\pi)$ measures the change in the utility of player 1 if he increases s_1 a small amount and player 2 responds optimally. On the other hand, $\frac{\partial u_2}{\partial s_1}$ measures the effect on player 2's utility from such a change. If the utilities of the players move in the same direction, then small changes that help player 1 will also help player 2. If the utilities move in opposite directions, then small changes that help player 1 will hurt player 2. The monotonicity of the externalities and best response functions ensures that this simple comparison is sufficient to determine if the game is a carrot or a stick game for a player.

The argument given in the last paragraph used an interior equilibrium to provide some intuition for the result. The proof of Proposition 3.1 does not rely on any of the equilibria being interior.

Using Proposition 3.1 we find that the quantity setting game of Example 3.1 is a stick game for each player while the price setting version is a carrot game for each player. We can then use Proposition 2.1 to obtain the well known result that sequential movement will be mutually beneficial in a price setting game, but not in a quantity setting game.

When one firm chooses quantities and the other chooses prices, as in Example 3.2, we find that the game is a stick game for the price setter and a carrot game for the quantity setter. Using Proposition 2.1 in this example we find that sequential movement will be mutually beneficial if the quantity setter

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moves ¯rst but not if the price setter moves ¯rst.

Furthermore, the price setter actually prefers to move second rather than ¯rst even though his reaction function is downward sloping. This is in contrast to Dowrick (Proposition 1, 1986) which states that if a ¯rm's reaction function slopes downward it will always prefer to move ¯rst rather than second. Dowrick's proposition requires that the externalities move in the same direction for both players, which is true if the goods are substitutes and the ¯rms use the same strategic variables as in Example 3.1.

In Example 3.2 the goods are no doubt substitutes, but the externalities move in opposite directions. Since this is a carrot game for the quantity setter, the price setter will bene¯t from sequential moves regardless of who moves ¯rst. Recall that moving ¯rst is always at least as good as simultaneous moves.

The ordering of pro¯ts for the price setter, ¯rm 2, is $\pi_2^F > \pi_2^L > \pi_2^S$, where the superscripts stand for (F)ollower, (L)eader and (S)imultaneous respectively. The exact values are: $\pi_2^F(\frac{3}{7}; \frac{11}{28}) = \frac{121}{784}$, $\pi_2^L(\frac{19}{42}; \frac{5}{14}) = \frac{175}{1176}$, and $\pi_2^S(\frac{18}{39}; \frac{5}{13}) = \frac{150}{1014}$.

The advertising game of Example 3.3 turns out to be a stick game for each player. Given the stick nature of the game, we know from Proposition 2.1 that sequential movement will never be mutually bene¯cial. A casual observation that the game involves strategic complements, however, might have led us erroneously to think otherwise.

The following Corollary, which is a direct result of Proposition 3.1, relates strategic complements and strategic substitutes to smooth monotonic carrot and stick games.

We call G a game with strategic substitutes whenever $b_1^0(s_2) < 0$ and $b_2^0(s_1) < 0$. We call G a game with strategic complements whenever $b_1^0(s_2) > 0$ and $b_2^0(s_1) > 0$.

Corollary 3.1: Let G be a smooth monotonic two person game.

a) If $\text{sign}[b_1^0(s_2)] = \text{sign}[b_2^0(s_1)]$ and $\text{sign}\left[\frac{\partial u_1}{\partial s_2}\right] = \text{sign}\left[\frac{\partial u_2}{\partial s_1}\right]$, then G is a game with strategic substitutes is equivalent to G is a stick game for each player, and G is a game with strategic complements is equivalent to G is a carrot game for each player.

b) If $\text{sign}[b_1^0(s_2)] = \text{sign}[b_2^0(s_1)]$ and $\text{sign}\left[\frac{\partial u_1}{\partial s_2}\right] \neq \text{sign}\left[\frac{\partial u_2}{\partial s_1}\right]$, then G is a game with strategic substitutes is equivalent to G is a carrot game for each player, and G is a game with strategic complements is equivalent to G is a stick game for each player.

The price and quantity setting games of Example 3.1 are examples of a) and the advertising game is an example of b).

4   Incentive Games

In Section 2 we discussed the implications of the carrot and stick nature of two person games for transformations of those games into games with sequential

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movements. The transformation from simultaneous moves to sequential moves involves changing only the information structure of the game.

In the current section we look at a transformation that affects the player set, the information structure of the game, and the payoffs. We start with a two player simultaneous move game and transform it to a four player two stage game. We call the new game an incentive game. The change to the information structure of the game comes from the introduction of the first stage. The second stage of the incentive game is identical to the original game in every respect except that the payoffs (incentives) of the players choosing there may be different. We concentrate on smooth monotonic games in this section.

One example of an incentive game is given by Sklivas (1987) (See also Fershtman and Judd (1987), Basu (1995), and Vickers (1984)). In Sklivas' example, owners of firms manipulate the incentives of their managers in an attempt to obtain higher profits.

Bonanno and Vickers (1988) and Rey and Stiglitz (1995) consider situations of vertical separation[2] where producers can write exclusive contracts with retailers and manipulate the incentives of the retailers in a way that yields the producers higher profits. These examples turn out also to be incentive games.

Baye, Crocker and Ju (1996) model horizontal separation of firms into multiple competing divisions. The incentives of the divisions are affected by the breakup. So essentially, these are much like incentive games. However, since we define incentive games to consist of only one division per firm, the propositions of this section do not directly apply. Yet, our results do help to describe what is going on in those situations.

We concentrate first on the example given by Sklivas (1987). Then we give some general results for incentive games.

Example 4.1 (Sklivas (1987)): There are two firms, each with an owner and a manager. The owner will choose the incentives for his manager. His manager will later choose a quantity to produce.[3] The managers compete in quantities in a market with linear demand and constant marginal cost. Each manager's remuneration is a linear combination of sales and profit. Hence, each is given incentive to maximize the following objective function: $O_i = (1 - \theta_i)\pi_i + \theta_i R_i$, where π_i and R_i denote the profit and revenue (sales) respectively of firm $i = 1, 2$, and θ_i is the strategy of the owner. By choosing θ_i different from zero, the owner can manipulate the incentives of his manager away from profit maximization. While this behavior would never be optimal in monopoly or perfect competition it may be in an oligopoly.

Letting C_i denote costs at firm i, we find that $O_i = R_i - (1 - \theta_i)C_i$. This shows that raising θ_i above zero is effectively making the manager act as if costs are lower than true costs. Lowering θ_i below zero makes the manager act as if costs are above true costs.

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[2]Vertical separation is just the reverse of vertical integration. An upstream firm and downstream firm vertically integrate into a merged firm, while a merged firm vertically separates into an upstream firm and a downstream firm.

[3]Sklivas also looked at price competition. Our results apply to that case also. We give only the quantity competition case here for ease of explanation.

After θ_1 and θ_2 have been chosen by the owners, each manager observes both his and his opponents incentives and then they compete in a quantity setting game.

Let the inverse demand function be given by $P(Q) = 1 - Q$ and cost at each firm by $C_i(q_i) = cq_i$. It is straightforward to derive that in the quantity setting game based on θ_1 and θ_2 the best response function of agent i will be $b_i(q_j) = \frac{1 - (1 - \theta_i)c - q_j}{2b}$.

Notice that raising θ_i will shift the best response function of that agent outward just as if marginal cost were reduced. Lowering θ_i shifts the best response function inward. The best response functions are similar to those in Figure 3.1.

Sklivas found that in the subgame perfect equilibrium of a two stage game where owners choose incentives in the first stage and managers compete in quantities in the second stage, the owners choose incentives that make the managers act like costs are less than true costs. By making his manager act like costs are lower, an owner induces him to produce more. Since the opposing manager can anticipate this over production by his rival, the hope is that the opposing manager will cut back output. However, since both managers are induced to overproduce, the total industry output rises and the profits at each firm fall.

We now show how to obtain an incentive game from a two player smooth monotonic game $G = [(S_1; S_2); (u_1; u_2)]$.

In Example 4.1, G is the game the managers play when each is given incentive to maximize the firm's profit, i.e., $\theta_1 = \theta_2 = 0$.

In this section we also assume that the utility functions $u_1(s_1; s_2)$ and $u_2(s_1; s_2)$ are twice continuously differentiable, and for each strategy choice of his opponent, a player's utility function is a strictly concave function of his own strategy. Strict concavity implies that $\frac{\partial^2 u_1(s_1; s_2)}{\partial s_1^2} < 0$ and $\frac{\partial^2 u_2(s_1; s_2)}{\partial s_2^2} < 0$ everywhere.

Incentive Game: $\Gamma = [(S_{p1}; S_{p2}; S_{a1}; S_{a2}); (u_{p1}(\cdot); u_{p2}(\cdot); u_{a1}(\cdot); u_{a2}(\cdot))]$ is a smooth monotonic stick/carrot nature preserving incentive game based on a two player smooth monotonic game $G = [(S_1; S_2); (u_1; u_2)]$ if and only if (4.1) to (4.9) are satisfied:

(4.1)(Principals Strategies) S_{p1} and S_{p2} are convex subsets of the real line R which include zero as an interior point.

(4.2)(Agents Strategies) $S_{a1} \subseteq \mathrm{ff}_1$ such that $f_1 : S_{p1} \times S_{p2} \to S_1\mathrm{g}$; and $S_{a2} \subseteq \mathrm{ff}_2$ such that $f_2 : S_{p1} \times S_{p2} \to S_2\mathrm{g}$, where S_1 and S_2 are both convex subsets of the real line R with the same smallest[4] element ω.

(4.3) (Principals Utility) for any strategy profile $(\theta_1; \theta_2; f_1; f_2)$, $u_{p1}(\theta_1; \theta_2; f_1; f_2) \equiv u_1(s_1; s_2)$ and $u_{p2}(\theta_1; \theta_2; f_1; f_2) \equiv u_2(s_1; s_2)$ where: $s_1 \equiv f_1(\theta_1; \theta_2)$ and $s_2 \equiv f_2(\theta_1; \theta_2)$.

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[4]Often in economics examples this smallest number will be taken to be zero. The existence of a smallest number is only used in the proof of Lemma 4.1 to show that the product of the slopes of the best response functions are not greater than 1.

(Agents Utility) We give the conditions for agent 1:

(4.4) (Smoothness and concavity) $u_1(\alpha_1; \alpha_2; f_1; f_2)$ is twice continuously differentiable and $\frac{\partial^2 u_{a1}}{\partial s_1^2} < 0$ everywhere.

(4.5)(Agent 1's Utility equals Principal 1's Utility if $\alpha_1 = 0$) for any strategy profile $(\alpha_1; \alpha_2; f_1; f_2)$ with $\alpha_1 = 0$, we have $u_{a1}(\alpha_1; \alpha_2; f_1; f_2) = u_{p1}(\alpha_1; \alpha_2; f_1; f_2)$.

(Effects of changing α): We give conditions for Principal/Agent 1

(4.6) (Each agent's best response function is independent of the incentives of the other agent) Let $b_1(s_2; \alpha_1)$ be the best response function of player 1 given $\alpha = (\alpha_1; \alpha_2)$. We require that $\frac{\partial b_1(s_2; \alpha_1)}{\partial \alpha_2} = 0$ everywhere.

(4.7) (raising α_1 harms Principal 2):
$\frac{\partial u_2(b_1(s_2; \alpha_1); s_2)}{\partial \alpha_1} - \frac{\partial u_2(s_1; s_2)}{\partial s_1} \frac{\partial b_1(s_2; \alpha_1)}{\partial \alpha_1} < 0$ for all s_2 and α_1.

(4.8) (Preservation of the stick/carrot nature of G): For each $\alpha = (\alpha_1; \alpha_2)$, in the subgame $G(\alpha) = ((S_1; S_2); (u_{a1}(\cdot); u_{2a}(\cdot)))$ played by the agents, the direction of the externalities and the slopes of the reaction functions are equivalent to those in G.

(4.9) For each $\alpha = (\alpha_1; \alpha_2)$ the best response function $b_1(s_2; \alpha_1)$ and $b_2(s_1; \alpha_2)$ satisfy $b_1'(s_2^*; \alpha_1)b_2'(s_1^*; \alpha_2) \leq 1$ whenever $s_1^* = b_1(s_2^*; \alpha_1)$ and $s_2^* = b_2(s_1^*; \alpha_2)$ are both met simultaneously.

In terms of Example 4.1, the game G where each manager maximizes its firm's profit is transformed into the four person incentive game Γ by turning each player into a team of two players, one principal (the owner) and one agent (the manager), and adding a preliminary stage before the game G is played. In this two stage game, the principals move simultaneously in the first stage. Each combination of the incentive choices $\alpha = (\alpha_1; \alpha_2)$ determines a subgame $G(\alpha)$ to be played by the agents in the second stage.

Condition (4.1) requires that each principal's strategy set includes zero as an interior point. Our intention is to use zero as the case when the principal gives his manager the same incentives as the corresponding player in the original game G. The choice variable of each principal affects the incentives of his manager and the strategy sets may differ across players. By condition (4.7) we assume that whenever principal 1 increases α_1 he induces his agent to act in a way that lowers the utility of the other principal.

Condition (4.2) just says that each agent's strategy describes what to do in each subgame $G(\alpha)$, and that the strategy set in each subgame is the same as the one used by the corresponding player in G. This condition connects the choices of the agent in Γ to those of the player in G.

Condition (4.3) requires that the payoff function of each principal be the same as the payoff function of the corresponding player in G. Condition (4.5) requires the same thing for agent i only in the case that his principal has chosen $\alpha_i = 0$. This condition connects the principal's and the agent's utility to that of the player in G.

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Condition (4.4) is a smoothness and concavity assumption on the payoff of each agent which will allow us to use the implicit function theorem to derive some comparative statics results about each agent's best response function.

Condition (4.6) says that each agent's best response function is not affected by the incentives of his rival. Nonetheless, the incentives of the rival agent will affect the payoff of an agent since it affects his rival's behavior.

As was mentioned earlier, condition (4.7) is that raising α_1 gives agent 1 incentives that lower the utility to principal 2.

Raising α_1 can be diagrammatically represented by shifting player 1's best response function in or out. Consider Figure 3.1. If player 1 inflicts a negative externality on player 2, then raising α_1 involves shifting his best response function out. In this case, for each strategy choice of player 2, player 1's best response involves a higher level of his strategy which by the negative externality implies player 2 will be worse off.

If alternatively player 1 offers a positive externality to player 2, then raising α_1 corresponds to shifting the best response function in. In Figure 3.1 this means the best response of player 1 is smaller for each level chosen by player 2. Notice that the shifts need not be parallel though they may be and in fact they are in the example of Sklivas (1987).

Condition (4.8) restricts incentive games by not allowing us to transform stick games into carrot games or vice versa.

Condition (4.9) will be used in the comparative statics results about changes in the equilibrium choices in the agents' subgame brought about by changes in α_1 or α_2.

A strategy combination $(\alpha_1; \alpha_2; f_1; f_2)$ is a subgame perfect equilibrium of incentive game Γ if and only if a) $(\alpha_1; \alpha_2; f_1; f_2)$ is a Nash equilibrium in Γ; and b) $f_1(\alpha) = s_1$ and $f_1(\alpha) = s_2$ define a Nash equilibrium in the subgame $G(\alpha)$ for each α.

All of the propositions of this section are made under the following assumption which we call assumption (A).

(A) Γ is a smooth monotonic stick/carrot nature preserving incentive game based on a smooth monotonic game G and $G(\alpha)$ has a unique Nash equilibrium which is interior for each α.

The assumption of uniqueness and an interior equilibrium has multiple important implications. Firstly, it implies that the subgame perfect equilibrium strategies of the agents, f_1 and f_2, are uniquely determined. Secondly, the assumption of an interior solution in a smooth monotonic game G implies by Proposition 3.1 that the game can be classified in terms of carrots and sticks for each player. By (4.8) the carrot/stick nature of the game will be the same in each subgame $G(\alpha)$. Lastly, it simplifies the comparative statics.

The following Lemmas are used in the proofs of the results. The first is an implication of the unique interior equilibrium in each subgame. The second lemma uses this implication to derive some comparative statics results.

Lemma 4.1 Let A and B be convex subsets of R each containing the same smallest element ε. Let $b_1 : A \to B$, and $b_2 : B \to A$ be differentiable functions. If the composite function $f(x) = b_1(b_2(x))$ has a unique fixed point x^π which is interior, then $f'(x^\pi) = b_1'(y^\pi)b_2'(x^\pi) \cdot 1$, where $y^\pi \doteq b_2(x^\pi)$.

Our intention is that $b_1(y)$ is the best response function of player 1 and $b_2(x)$ is the best response function of player 2. Whenever x^π is a fixed point of $f(x^\pi)$ it follows that (x^π, y^π) is a Nash equilibrium. Lemma 4.1 states that if a Nash equilibrium is unique and interior, then the product of the slopes of the best response functions must be no greater than 1 at the equilibrium point.[5] This result is used to obtain the comparative statics results presented in Lemma 4.2. We give results for changing α_1.

Lemma 4.2: Under assumption (A), let $(\alpha_1, \alpha_2, f_1, f_2)$ be a strategy for which $(f_1(\alpha'), f_2(\alpha'))$ is a Nash equilibrium in $G(\alpha')$ for each $\alpha' \in S_{p1} \times S_{p2}$. Then:

(4.10) $\operatorname{sign}\left[\dfrac{\partial f_1(\alpha)}{\partial \alpha_1}\right] \leqq \operatorname{sign}\left[\dfrac{\partial u_2}{\partial s_1}\right]$; and

(4.11) $\operatorname{sign}\left[\dfrac{\partial f_2(\alpha)}{\partial \alpha_1}\right] \leqq \operatorname{sign}\left[\dfrac{\partial u_1}{\partial s_2}\right]$ if G is a carrot game for player 1, and $\operatorname{sign}\left[\dfrac{\partial f_2(\alpha)}{\partial \alpha_1}\right] = \operatorname{sign}\left[\dfrac{\partial u_1}{\partial s_2}\right]$ if G is a stick game for player 1.

(4.12) $\operatorname{sign}\left[\dfrac{\partial b_1(s_2, \alpha_1)}{\partial \alpha_1}\right] \leqq \operatorname{sign}\left[\dfrac{\partial u_2}{\partial s_1}\right]$

In Lemma 4.2 we connect the effect of changing α_1 in the incentive game to the payoff functions $u_1(s_1, s_2)$ and $u_2(s_1, s_2)$ in the original game G. We can do this since the restriction that the stick/carrot nature is preserved (4.8) implies that the direction of the externalities is preserved.

Whenever we write that player 1 is better off in Γ than in G we mean that in any subgame perfect equilibrium of Γ, principal 1's utility is higher than player 1's utility in the unique Nash equilibrium of G. Recall that principal 1 and player 1 have the same utility so we can think of them as the same player and compare their utilities across games.

The first result for incentive games is that the carrot or stick behavior of each agent will be magnified (Proposition 4.1). If G is a stick game for a player, then the agent of that player will be induced to act more harmfully in Γ, that is, he will behave more stick like. Alternatively, if G is a carrot game for a player, then his agent will behave more carrot like in Γ. In this sense, the incentive game tends to magnify the carrot or stick behavior of a player.

Proposition 4.1: Under assumption (A), the carrot or stick behavior of each player is magnified in the incentive game, i.e., for any subgame perfect equilibrium $(\alpha_1^\pi, \alpha_2^\pi, f_1^\pi, f_2^\pi)$:

If G is a carrot game for player 1, then $\alpha_1^\pi < 0$; and

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[5] This result could have been obtained by the weaker assumption that only one player has a minimum or a maximum strategy. Often the result of Lemma 4.1 is assumed. I was curious when it could be derived.

If G is a stick game for player 1, then $\alpha_1^\sharp > 0$.

We can use Proposition 4.1 to show that if the original game is a carrot game for a player, then his opponent will be better o® in the incentive game. This result is rather intuitive since more carrot like behavior will surely bene¯t an opponent.

Proposition 4.2: Under assumption (A), if G is a carrot game for player 1, then player 2 will be better o® in ¡ than in G.

Note that this proposition does not depend on the original game being a carrot game for player 2. Indeed, player 2 will bene¯t in the incentive game even if the original game is a stick game for him. In the price/quantity setting example of Section 3 we can predict that the price setter will de¯nitely bene¯t from any incentive game. This comes from the fact that quantity setters are carrot players in a price/quantity game and carrot players help their opponents in incentive games. Whether or not the quantity setter gains is ambiguous, since there will be o®setting e®ects.

Applying Proposition 4.2 to a game which is a carrot game for each player we observe that both players will be better o® in the incentive game. So, for example, incentive games based on price competition will bene¯t both ¯rms.

Corollary 4.1 Under assumption (A) if G is a carrot game for each player, then both are better o® in ¡ than in G.

In Corollary 4.1, the only type of symmetry we require is that G is a carrot game for each player. The players may have di®erent strategy spaces and di®erent payo® functions.

Sklivas (1987) looked at both price and quantity competition in symmetric games. We de¯ne symmetry later, but basically a game is symmetric if the players are a priori identical up to their names. Sklivas found that under price competition both players were better o® in the incentive game and under quantity competition both players were worse o®.

The result for price competition follows directly from Corollary 4.1 and requires only that the game is a carrot game for each player. The symmetry used by Sklivas about the payo® functions and strategy sets is not necessary. The result for quantity competition, however, requires such symmetry between the players. Without enough symmetry, e.g., if the players have di®erent payo®s or strategies, the result that both are worse o® may not hold.

In general, if the original game is a stick game for each player, we know by Proposition 4.1 that each principal will magnify the incentives of his agent to harm the other principal. This implies that at least one player will be worse o®.

Proposition 4.3 Under assumption (A), if G is a stick game for each player, then at least one player is worse o® in ¡ than in G.

To obtain the result that both players are worse off we should assume more symmetry between the players than simply that both are stick players. We use the following definition of symmetry given in Gal-Or (1985).

A two player game $G = ((S_1; S_2); (u_1; u_2))$ is symmetric if and only if $S_1 = S_2$, and $u_1(s_1; s_2) = u_2(s_2; s_1)$ for all $s_1; s_2 \in S_1$.

This definition of symmetry captures the idea that the two players are a priori identical up to their names. The price setting and quantity setting games of Example 3.1 are symmetric. The advertising game (Example 3.3), and the mixed price/quantity competition (Example 3.2) are not symmetric, nor is Example 4.2.

Based on the above definition of symmetry we define an incentive game Γ to be a symmetric incentive game if and only if a) $S_{p1} = S_{p2}$, $S_{a1} = S_{a2}$, and b) $u_{p1}(\alpha_1; \alpha_2; f_1; f_2) = u_{p2}(\alpha_2; \alpha_1; f_2; f_1)$ and $u_{a1}(\alpha_1; \alpha_2; f_1; f_2) = u_{a2}(\alpha_2; \alpha_1; f_2; f_1)$ for all $(\alpha_1; \alpha_2; f_1; f_2)$.

This restriction requires that the game G is symmetric and also that the two principal-agent teams are identical a priori up to their names.

As was mentioned, the examples studied by Sklivas are symmetric incentive games.

In a symmetric game we often concentrate on symmetric equilibria. In an incentive game Γ an equilibrium $(\alpha_1; \alpha_2; f_1; f_2)$ is symmetric if $\alpha_1 = \alpha_2$ and $f_1 = f_2$. We have the following result for stick games which corresponds to Corollary 4.1 for carrot games. Notice the use of symmetry in this result.

Corollary 4.2 Under assumption (A) if Γ is a symmetric incentive game and G is a stick game for each player, then in any symmetric subgame perfect equilibrium of Γ both players are worse off than in G.

Corollary 4.2 implies that both players are worse off in a symmetric incentive game based on quantity competition. The next example is an asymmetric quantity setting game where one player has a cost advantage. We find that the player with the cost advantage is better off in the unique subgame perfect equilibrium of the incentive game.

Example 4.2: Demand is linear and given by $P(q_1 + q_2) = 1 - (q_1 + q_2)$, and marginal costs are constant at a firm and given by $c_1 = .65$ and $c_2 = .5$ for firms 1 and 2 respectively. The choice of an owner i is a parameter α_i which is the weight placed on the profits in his manager's objective function. Manager i's objective will be to maximize $O_i = R_i - (1 - \alpha_i)c_i q_i$.

For each $(\alpha_1; \alpha_2)$ there is a unique equilibrium in the second stage game. We concentrate on equilibria where the quantities are positive for both firms. Then it is straightforward to show that the equilibrium profits are given by:

$$\pi_i(\alpha_1; \alpha_2) = \frac{1}{9}[1 + (1 - \alpha_j)c_j - (2 + \alpha_i)c_i][1 + (1 - \alpha_j)c_j - 2(1 - \alpha_i)c_i] \text{ for } i = 1; 2; \text{ and } j \neq i.$$

An interior equilibrium in the preliminary stage requires that: $\frac{\partial \pi_i(\alpha_1; \alpha_2)}{\partial \alpha_i} = 0$ for $i = 1; 2$. This implies that the equilibrium $\alpha_i^\times$'s are given by:

$$1 - \alpha_i^\times = \frac{8c_i - 1 - 2c_j}{5c_i}, \text{ for } i = 1; 2 \text{ and } j \neq i.$$

For the given values $c_1 = .65$ and $c_2 = .5$, we find that $\alpha_1^* = \frac{5}{325} > 0$, and $\alpha_2^* = \frac{8}{25} > 0$. Hence, both players use more harmful incentives. The equilibrium quantities are given by $q_i^* = \frac{1 - 2(1-\alpha_i)c_i + (1-\alpha_j)c_j}{3}$ for $i = 1, 2$ and $j \neq i$. From this it is straightforward to show that in moving from the original game G to the incentive game $G(\alpha)$, the equilibrium quantity and profits of firm 1 fall and the equilibrium quantity and profits of firm 2 rise. The firm with the cost advantage is made slightly better off even though both firms act more harmfully. The actual values are $q_1^G = .067$, $q_2^G = .2167$, $\pi_1^G = .044$, $\pi_2^G = .0469$, $q_1^* = .02$, $q_2^* = .32$, $\pi_1^* = .0002$, $\pi_2^* = .0512$, where the superscripts G and $*$ stand for the original game and incentive game respectively.

Intuitively, more harmful incentives have a cost which is determined here by the cost of output. For firm 1 the cost of bashing his opponent with a stick is higher than for firm 2 since his marginal cost is higher. This cost advantage leads firm 2 ultimately to bash his opponent more and firm 1 to bash less.

We now show that the example of vertical separation given in Bonanno and Vickers (1988) is an incentive game.[6]

Example 4.3: Two producers make imperfect substitutes and compete in prices. Each hires a retailer to sell its product and the producer can extract all his retailer's profit with a franchise fee.[7] The producer also chooses a per unit wholesale price w_i to charge his retailer, which may or may not differ from the true marginal cost. The retailers then compete in retail prices taking each others wholesale prices as given.

In the terminology of incentive games, the original game is defined to be the one the retailers play when wholesale prices are chosen to equal marginal cost. In the first stage of the incentive game, each producer i chooses a wholesale price w_i, which determines the incentives of his retailer. In the second stage, each retailer chooses a retail price p_i to maximize its profit which will differ from the firm's true profit when wholesale price differs from marginal cost.

Let demand at retailer i be given by $D_i(p_1, p_2) = A - bp_i + dp_j$, for $i = 1, 2$ ($j \neq i$), where p_1 and p_2 are the retail prices. Marginal cost at each producer firm is a constant $c > 0$. The profit to retailer i is: $\pi_i^R(p_1, p_2) = (p_i - w_i)D_i(p_1, p_2)$, since by assumption it incurs no cost other than the wholesale price. This leads to the best response function: $b_i(p_j, w_i) = \frac{(A + bw_i)}{2b} + \frac{dp_j}{2b}$. To get this in a form to use the propositions of this section we define $\alpha_i = c - w_i$, so that the true incentives are given when $w_i = c$, i.e., $\alpha_i = 0$. Then clearly raising α_i is done by lowering wholesale price w_i below marginal cost. Alternatively, lowering α_i is done by raising wholesale price w_i above marginal cost. Each principal i (producer) will choose α_i to maximize the true profit of firm i, $\pi_i = (p_i - c)D_i(p_1, p_2)$ since it can extract the retailer's profit by use of a fixed franchise fee.

The game played by the retailers is a carrot game for each pair (α_1, α_2). Applying Corollary 4.1 we immediately obtain the result of Bonanno and Vick-

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[6] So is the example of Rey and Stiglitz (1995) when franchise fees are allowed.

[7] If a franchise fee is not allowed, then the objective of the producer may not be to maximize original profits and the propositions of this section cannot be applied.

ers (1988) that both producers are made better o® from vertical separation.

Baye, Crocker and Ju (1996) analyze horizontal separation. A ¯rm horizon-
tally separates by breaking up into multiple competing divisions. The authors
start with two ¯rms competing in quantities so it is a stick game for each player.
They then allow each ¯rm to break up into multiple competing divisions in an
previous stage. The e®ect of this breakup is to make the broken up ¯rms more
harmful to their rivals than before. While the propositions of this section do
not directly apply to this example, the spirit does. Since the original game is
a stick game for each player each will tend to do more harmful things in the
two stage game. Breaking up into multiple divisions is more harmful. Since the
players are symmetric, it turns out that they are both worse o® after the break
up.

5   Relationship to Stackelberg Solvable Games

Stick games appear to be similar in some respects to zero sum games and carrot
games appear to be similar to coordination games. In this section we show that
Stackelberg solvable games, which include zero sum games and coordination
games, are neither carrot nor stick games. Hence, to make use of the distinction
between carrots and sticks we must go beyond Stackelberg solvable games.

D'Aspremont and Gerard-Varet (1986) introduced the notion of Stackelberg
solvable games. They were interested in games where changes to the order of
moves or the incentives of agents, do not a®ect the outcome of the game.

A game G is called Stackelberg solvable if it possesses at least one Nash
equilibrium $(s_1; s_2)$ such that:

(5.1) s_1 maximizes $u_1(s_1; b_2(s_1))$, and
(5.2) s_2 maximizes $u_2(b_1(s_2); s_2)$.

It follows directly from the de¯nition that ∇_1 and ∇_2 in a Stackelberg solvable
game G are determined by a pair $(s_1; s_2)$ which satis¯es (5.1) and (5.2).

Stackelberg solvable games include both zero sum games and coordination
games. They also include games like the prisoners' dilemma which is neither
zero-sum nor a coordination game. Figure 5.1 is an example of a zero sum game
and Figure 5.2 is an example of a coordination game.

The game of Figure 5.1 has a unique Nash equilibrium (T,L) and thus
$\underline{v}_1 = \nabla_1 = 5$, and $\underline{v}_2 = \nabla_2 = ¡5$. The game of Figure 5.2 has two Nash
equilibria, (T,L) and (B,R). Since they can be Pareto ranked, we ¯nd that $\underline{v}_1 = \underline{v}_2 = 1$, and $\nabla_1 = \nabla_2 = 2$.

		Figure 5.1	
		Player 2	
		L	R
Player 1	T	5,-5	10,-10
	B	3,-3	-2,2

		Figure 5.2	
		Player 2	
		L	R
Player 1	T	2,2	-1,-1
	B	0,0	1,1

In a zero sum game, the players' objectives are diametrically opposed while in coordination games they are perfectly aligned. In this sense, these games are at opposite extremes. However, if we allow pre-play communication, but no pre-play commitment to strategies, then zero sum games and coordination games share a similarity in that each player is indifferent between: the simultaneous move game, the sequential move game where player 1 moves first, and the sequential move game where player 2 moves first.

Sequential moves might allow the player moving first to: a) commit to a strategy, and b) choose a Pareto optimal Nash equilibrium strategy. Pre-game communication eliminates the value of b). This allows us to concentrate on a). We do not mean to imply that b) is not important. In fact b) is of no value to players in zero sum games, while players in a coordination game with multiple equilibria may find great value in b). Pre-game communication solves the problem of coordinating on the best equilibrium and lets us concentrate on the value of commitment.

That the players will be indifferent between the three situations in a coordination game seems rather trivial if they can communicate beforehand. Since the Nash equilibria in a coordination game can be Pareto ranked, the players will no doubt choose strategies that yield the maximum Nash equilibrium payoff of the simultaneous move game. In either of the sequential move versions of the original game, the player moving first will be compelled to choose the strategy that leads to his maximum Nash equilibrium payoff and his opponent will follow suit. It is because of the alignment of objectives and pre-play communication that sequential movements are of no value.

The argument that a) has no value for two person zero sum games is not so straightforward. It is a result of the minimax theorem. Von Neumann and Morgenstern discuss this result in their book \Theory of Games and Economic Behavior" (Section 14.2). In a two person zero sum game, moving second \appears" to be better since you can always find out your opponents strategy and minimize against it. However, it turns out that there is no difference between moving first, second, or simultaneously when an equilibrium exists.

We now give the main result of this section.

Proposition 5.1. If a game is Stackelberg solvable, then it is neither a carrot game nor a stick game for either player.

6   Conclusions

Carrots and sticks have provided insights into whether or not both players will benefit from transformations to games with sequential movements or to incentive games.

We showed in Section 2 that sequential movement is mutually beneficial in carrot games and unilaterally beneficial in stick games (Proposition 2.1).

In Section 3 we defined smooth monotonic games. We showed every such game is either a carrot or a stick game for each player provided the Nash equilibria are interior (Proposition 3.1). We also compared carrot and stick games to

the notions of strategic substitutes and strategic complements (Corollary 3.1). We used an advertising game (Example 3.3) to show how standard expectations about strategic complements and strategic substitutes might lead one astray.

In Section 4 we used the results obtained for smooth monotonic games to obtain some further results for incentive games. Specifically, we showed that if a game is a carrot game for one player, his opponent will benefit in any incentive game based on the original game (Proposition 4.1). This leads to the result that incentive games based on carrot games for both players generate outcomes which are mutually beneficial (Corollary 4.1). For incentive games based on stick games, the results were weaker. The corresponding result to Proposition 4.1 that a stick player will make his opponent worse off in an incentive game was not obtained. In Proposition 4.3 we showed that incentive games based on a stick game for each player will harm at least one player, but not necessarily both.

7 Appendix

Proof of Proposition 2.1: Let $(s_1^*; f_2^*)$ be a subgame perfect equilibrium in G^0.

If G is a carrot game then there exists a $s_1 \in S_1$ such that $u_1(s_1; b_2(s_1)) > \bar{v}_1$. Then $u_1(s_1^*; f_2^*(s_1^*)) = u_1(s_1^*; b_2(s_1^*)) > \bar{v}_1$ since $(s_1^*; f_2^*)$ is a subgame perfect equilibrium. Thus by (2.1) it follows that $u_2(s_1^*; f_2^*(s_1^*)) > \bar{v}_2$.

If G is a stick game then we find by similar reasoning using (2.2) and the definition of a subgame perfect equilibrium that $u_1(s_1^*; f_2^*(s_1^*)) > \bar{v}_1$ and $u_2(s_1^*; f_2^*(s_1^*)) < \underline{v}_2$. ¤

Proof of Lemma 2.1: First consider the case where $\bar{v}_1$ and $\bar{v}_2$ are defined by the same Nash equilibrium $(s_1; s_2)$. Suppose that G is a carrot or a stick game for player 1. Then we show that G cannot be a carrot or a stick game for player 2.

By the assumption about player 1, there exists a $s_1^0 \neq s_1$ such that $u_1(s_1^0; b_2(s_1^0)) > \bar{v}_1 = u_1(s_1; s_2)$. Since this is a 2x2 game it follows that s_1^0 is the only other strategy for player 1. Also, $b_2(s_1^0) \neq s_2$, since if $b_2(s_1^0) = s_2$, then we have $u_1(s_1^0; s_2) > u_1(s_1; s_2)$ which contradicts $(s_1; s_2)$ is a Nash equilibrium. So $b_2(s_1^0) \; \hat{} \; s_2^0$ is the only other strategy of player 2. For G to be a carrot or stick game for player 2 we require that $u_2(b_1(s_2^0); s_2^0) > \bar{v}_2 = u_2(s_1; s_2)$. If $b_1(s_2^0) = s_1^0$, then $(s_1^0; s_2^0)$ is a Nash equilibrium which violates the assumption that the value $\bar{v}_2$ is defined by $(s_1; s_2)$. Alternatively, if $b_1(s_2^0) = s_1$, then $u_2(s_1; s_2^0) > u_2(s_1; s_2)$ which contradicts $(s_1; s_2)$ is a Nash equilibrium. In either case we have come to a contradiction, so G cannot be a carrot or stick game for player 2.

Next, suppose that $\bar{v}_1 = u_1(s_1; s_2)$ and $\bar{v}_2 = (s_1^0; s_2^0)$ and $(s_1; s_2) \neq (s_1^0; s_2^0)$. Then it follows that both $s_1 \neq s_1^0$ and $s_1^0 \neq s_2^0$. If $s_1 = s_1^0$, then $b_2(s_1)$ is unique valued implies that $s_2 = s_2^0$. Similarly, if $s_2 = s_2^0$, then $b_1(s_2)$ is unique valued implies that $s_1 = s_1^0$. Since $(s_1^0; s_2^0)$ is a Nash equilibrium and best responses are unique, $b_1(s_2^0) = s_1^0$, and $b_2(s_1^0) = s_2^0$. Since s_1 defines $\bar{v}_1$; and s_1^0 is the only other strategy for player 1, it follows that $u_1(s_1^0; b_2(s_1^0)) \cdot \bar{v}_1$, and thus G is neither a carrot nor stick game for player 1. A similar argument holds for player 2. ¤

The following lemma is used in the proof of Proposition 3.1.

Lemma 3.1. Let G be a smooth monotonic two player game and let $(s_1^n; s_2^n)$ be a Nash equilibrium in G.

(3.6) If $\left[\frac{\partial u_1}{\partial s_2} b_2'(s_1)\right] > 0$; then $u_1(s_1; b_2(s_1)) > u_1(s_1^n; s_2^n)$ implies $s_1 > s_1^n$.

(3.7) If $\left[\frac{\partial u_1}{\partial s_2} b_2'(s_1)\right] < 0$; then $u_1(s_1; b_2(s_1)) > u_1(s_1^n; s_2^n)$ implies $s_1 < s_1^n$.

Proof of Lemma 3.1: We prove the contrapositive of the part of (3.6) following the comma. That is, we prove under the assumption $\left[\frac{\partial u_1}{\partial s_2} b_2'(s_1)\right] > 0$ that $[not(s_1 > s_1^n)]$ implies $[not(u_1(s_1; b_2(s_1)) > u_1(s_1^n; s_2^n))]$. The contrapositive of the corresponding part of (3.7) can be proved in a similar manner.

Since $s_1 = s_1^n$ implies $u_1(s_1; b_2(s_1)) = u_1(s_1^n; s_2^n)$ we can concentrate on the case when $s_1 < s_1^n$. Suppose $s_1 < s_1^n$. By assumption $\left[\frac{\partial u_1}{\partial s_2} b_2'(s_1)\right] > 0$, either $\frac{\partial u_1}{\partial s_2} < 0$ and $b_2'(s_1) < 0$ or $\frac{\partial u_1}{\partial s_2} > 0$ and $b_2'(s_1) > 0$. In the first case: $u_1(s_1; b_2(s_1)) < u_1(s_1; s_2^n) \cdot u_1(s_1^n; s_2^n)$ where the first inequality follows since $\frac{\partial u_1}{\partial s_2} < 0$ and $b_2(s_1) > s_2^n$, and the second inequality follows from utility maximization of the player 1. The second case is proved in the same way. ¤

Proof of Proposition 3.1: In the following consider an arbitrary s_1 chosen such that $u_1(s_1; b_2(s_1)) > \nabla_1$ and let $(s_1^n; s_2^n)$ satisfy $u_2(s_1^n; s_2^n) = \nabla_2$ and $u_2(s_1^0; s_2^0) = \underline{v}_2$. Then $u_1(s_1; b_2(s_1)) > u_1(s_1^n; s_2^n)$ and $u_1(s_1; b_2(s_1)) > u_1(s_1^0; s_2^0)$. In the following we fix such $s_1; b_2(s_1); s_1^n; s_2^n; s_1^0; s_2^0$.

The If-part of (3.4) is true only if $\left[\frac{\partial u_2}{\partial s_1} > 0 \text{ and } sign\left[\frac{\partial u_1}{\partial s_2}\right] = sign\left[b_2'(s_1)\right]\right]$ or $\left[\frac{\partial u_2}{\partial s_1} < 0 \text{ and } sign\left[\frac{\partial u_1}{\partial s_2}\right] \neq sign\left[b_2'(s_1)\right]\right]$. In the first case we have by Lemma 3.1 (3.6) that $u_1(s_1; b_2(s_1)) > u_1(s_1^n; s_2^n)$ implies $s_1 > s_1^n$. But then $u_2(s_1^n; s_2^n) < u_2(s_1; s_2^n) \cdot u_2(s_1; b_2(s_1))$ where the first inequality follows since $s_1 > s_1^n$ and $\frac{\partial u_2}{\partial s_1} > 0$ and the second inequality follows from utility maximization of player 1. In the second case we have by Lemma 3.1 (3.7) that $u_1(s_1; b_2(s_1)) > u_1(s_1^n; s_2^n)$ implies $s_1 < s_1^n$. But then $u_2(s_1^n; s_2^n) < u_2(s_1; s_2^n) \cdot u_2(s_1; b_2(s_1))$ where the first inequality follows since $s_1 < s_1^n$ and $\frac{\partial u_2}{\partial s_1} < 0$ and the second inequality follows by utility maximization of player 2.

The If-part of statement (3.5) is true only if $\left[\frac{\partial u_2}{\partial s_1} < 0 \text{ and } sign\left[\frac{\partial u_1}{\partial s_2}\right] = sign\left[b_2'(s_1)\right]\right]$ or $\left[\frac{\partial u_2}{\partial s_1} > 0 \text{ and } sign\left[\frac{\partial u_1}{\partial s_2}\right] \neq sign\left[b_2'(s_1)\right]\right]$.

In the first case we have by Lemma 3.1 (3.6) that $u_1(s_1; b_2(s_1)) > u_1(s_1^0; s_2^0)$ implies $s_1 > s_1^0$. But then $u_2(s_1; b_2(s_1)) < u_2(s_1^0; b_2(s_1)) \cdot u_1(s_1^0; s_2^0)$ where the first inequality follows from $s_1 > s_1^0$ and $\frac{\partial u_2}{\partial s_1} < 0$ and the second inequality follows from utility maximization of player 2. The result in the second case can be proved in the same manner. ¤

Proof of Lemma 4.1: Consider the differentiable, and thus continuous, one to one function $g(x) = x$ defined from B onto B. At the unique fixed point x^α we have $f(x^\alpha) = g(x^\alpha)$. Furthermore, since x^α is a unique fixed point of

$f(¢)$ and $f(¢)$ is continuous it follows that for all $x < x^{¤}$, $g(x) < f(x)$. If there was an $x < x^{¤}$, with $g(x) ¸ f(x)$, then since $g(") = "$, and $f(") ¸ "$, it would follow by the intermediate value theorem that there is an $x^{0} \in ["; x]$ such that $f(x^{0}) = g(x^{0}) = x^{0}$. But this contradicts the uniqueness of the ¯xed point $x^{¤}$.

Since $g(x) < f(x)$ for all $x < x^{¤}$ and $g(x^{¤}) = f(x^{¤})$ and $x^{¤}$ is interior, we have that for all $± < 0$ in some neighborhood of $x^{¤}$, $\frac{g(x^{¤}+±) ¡ g(x^{¤})}{±} > \frac{f(x^{¤}+±) ¡ f(x^{¤})}{±}$. Taking the limit as $± ! 0$ we obtain $g^{0}(x^{¤}) ¸ f^{0}(x^{¤})$. But $g^{0}(x^{¤}) = 1$, and $f^{0}(x^{¤}) = b_1^{0}(y^{¤}) b_2^{0}(x^{¤})$. ¤

Proof of Lemma 4.2: The ¯rst order conditions are:

(A.1) $\frac{@u_{a1}(®_1;®_2;s_1;s_2)}{@s_1} = 0$, and

(A.2) $\frac{@u_{a2}(®_1;®_2;s_1;s_2)}{@s_2} = 0$.

By assumption that an interior equilibrium exists for each $®$ we have solutions to (A.1) and (A.2) for all $(®_1;®_2;s_1;s_2)$. By assumption (4.4) we can use the implicit function theorem on either one of the ¯rst order conditions separately, e.g., to ¯nd the slope of the best response function, since by strict concavity $\frac{@^2 u_{a1}}{@s_1^2} < 0$ and $\frac{@^2 u_{a2}}{@s_2^2} < 0$ (See, e.g., Fulks (1978)).

To use the implicit function theorem on both equations simultaneously to calculate for example $\frac{@f_1(®)}{@®_1}$ requires that: $D = \frac{@^2 u_{a1}}{@s_1^2}\frac{@^2 u_{a2}}{@s_2^2} ¡ \frac{@^2 u_{a1}}{@s_2 @s_1}\frac{@^2 u_{a2}}{@s_1 @s_2} \neq 0$ at s_1 and s_2 that satisfy (A.1) and (A.2) simultaneously. Using the implicit function theorem on the ¯rst order conditions independently we ¯nd that:

$b_1^{0}(s_2;®_1) = ¡\left[\frac{\frac{@^2 u_{a1}}{@s_2 @s_1}}{\frac{@^2 u_{a1}}{@s_1^2}}\right]$ and $b_2^{0}(s_1;®_2) = ¡\left[\frac{\frac{@^2 u_{a2}}{@s_1 @s_2}}{\frac{@^2 u_{a2}}{@s_2^2}}\right]$. Substituting these values into D and rearranging terms we obtain:

$D = \frac{@^2 u_{a1}}{@s_1^2}\frac{@^2 u_{a2}}{@s_2^2}[1 ¡ b_1^{0}(s_2;®_1) b_2^{0}(s_1;®_2)]$.

By Lemma 4.1 and (4.9) we have that $b_1^{0}(s_2;®_1) b_2^{0}(s_1;®_2) < 1$ at s_1 and s_2 that satisfy both (A.1) and (A.2) simultaneously. This together with assumption (4.4) that the utility functions are strictly concave implies $D > 0$.

From the ¯rst order condition of player 1 we ¯nd using the implicit function theorem that:

(A.3) $\frac{@b_1(s_2;®_1)}{@®_1} = ¡\left[\frac{\frac{@^2 u_{a1}}{@®_1 @s_1}}{\frac{@^2 u_{a1}}{@s_1^2}}\right]$.

By assumption (4.7) that $\frac{@u_2(b_1(s_2;®_1);s_2)}{@®_1} - \frac{@u_2}{@s_1}\frac{@b_1(s_2;®_1)}{@®_1} < 0$, it follows that:

(A.4) sign $\frac{@b_1(s_2;®_1)}{@®_1} \neq$ sign $\frac{@u_2}{@s_1}$.

Thus we have shown (4.12). By this result and (A.3) and strict concavity of the utility function we have that:

(A.5) sign $\frac{@^2 u_{a1}}{@®_1 @s_1} \neq$ sign $\frac{@u_2}{@s_1}$.

Using the implicit function theorem on both ¯rst order conditions we get:

(A.6) $\dfrac{\partial f_1(\textcircled{R})}{\partial \textcircled{R}_1} = \dfrac{\left(\dfrac{\partial^2 u_{a1}}{\partial \textcircled{R}_1 \partial s_1}\right)\left(-\dfrac{\partial^2 u_{a2}}{\partial s_2^2}\right)}{D}$, and

(A.7) $\dfrac{\partial f_2(\textcircled{R})}{\partial \textcircled{R}_1} = \dfrac{\left(\dfrac{\partial^2 u_{a1}}{\partial \textcircled{R}_1 \partial s_1}\right)\left(\dfrac{\partial^2 u_{a2}}{\partial s_1 \partial s_2}\right)}{D} = \dfrac{\partial f_1(\textcircled{R})}{\partial \textcircled{R}_1} b_2^0(s_1; \textcircled{R}_1)$.

Since $D > 0$ and $-\dfrac{\partial^2 u_{a2}}{\partial s_2^2} > 0$, it follows by (A.5) and (A.6) that:

(A.8) $\text{sign}\left(\dfrac{\partial f_1(\textcircled{R})}{\partial \textcircled{R}_1}\right) = \text{sign}\left(\dfrac{\partial u_{a2}}{\partial s_1}\right)$.

We have proved (4.10). It also follows that:

(A.9) $\text{sign}\left(\dfrac{\partial f_2(\textcircled{R})}{\partial \textcircled{R}_1}\right) = \text{sign}\left(\dfrac{\partial u_{a2}}{\partial s_1}\right)$ if and only if $b_2^0(s_1; \textcircled{R}_1) > 0$.

To get to (4.11) we apply the results of Proposition 3.1 for smooth monotonic carrot and stick games.

For a stick game for player 1, $-\text{sign}\left[\dfrac{\partial f_1(\textcircled{R})}{\partial \textcircled{R}_1}\right] = \text{sign}\left[\dfrac{\partial u_{a2}}{\partial s_1}\right] = -\text{sign}\left[\dfrac{\partial u_{a1}}{\partial s_2} b_2^0(s_1; \textcircled{R}_1)\right]$.

The last term equals $-\text{sign}\left[\dfrac{\partial u_{a1}}{\partial s_2}\right]$ if and only if $b_2^0(s_1; \textcircled{R}_1) > 0$. From this we obtain the result (4.11) for stick games. The result for carrot games is proved in the same way. ¤

Proof of Proposition 4.1: Let $(\textcircled{R}_1^{\text{¤}}; \textcircled{R}_2^{\text{¤}}; f_1; f_2)$ be given such that $(f_1(\textcircled{R}); f_2(\textcircled{R}))$ is a Nash equilibrium in $G(\textcircled{R})$ for each $\textcircled{R} \in S_{p1} \times S_{p2}$. Let $(s_1^{\text{¤}}; s_2^{\text{¤}})$ denote the unique Nash equilibrium in $G(\textcircled{R}^{\text{¤}})$. By assumption (4.3)) we have that $u_{p1}(\textcircled{R}_1; \textcircled{R}_2; f_1; f_2) = u_1(s_1; s_2)$, and thus:

(A.10) $\dfrac{\partial u_{p1}(\textcircled{R}_1^{\text{¤}}; \textcircled{R}_2^{\text{¤}}; f_1; f_2)}{\partial \textcircled{R}_1} = \dfrac{\partial u_1(s_1^{\text{¤}}; s_2^{\text{¤}})}{\partial s_1} \dfrac{\partial f_1(\textcircled{R}^{\text{¤}})}{\partial \textcircled{R}_1} + \dfrac{\partial u_1(s_1^{\text{¤}}; s_2^{\text{¤}})}{\partial s_2} \dfrac{\partial f_2(\textcircled{R}^{\text{¤}})}{\partial \textcircled{R}_1}$.

First we show that if $\textcircled{R}_1^{\text{¤}} > 0$ then $\dfrac{\partial u_1(s_1^{\text{¤}}; s_2^{\text{¤}})}{\partial s_1} \dfrac{\partial f_1(\textcircled{R}^{\text{¤}})}{\partial \textcircled{R}_1} < 0$, and if $\textcircled{R}_1^{\text{¤}} < 0$ then $\dfrac{\partial u_1(s_1^{\text{¤}}; s_2^{\text{¤}})}{\partial s_1} \dfrac{\partial f_1(\textcircled{R}^{\text{¤}})}{\partial \textcircled{R}_1} > 0$.

Let $s_1^0 = b_1(s_2^{\text{¤}}; 0)$. Then by strict concavity of $u_1(s_1; s_2)$ for each given s_2 we have that $\dfrac{\partial u_1(s_1^0; s_2^{\text{¤}})}{\partial s_1} = 0$, and $\dfrac{\partial u_1(s_1; s_2^{\text{¤}})}{\partial s_1} < 0$ for all $s_1 > s_1^0$, and $\dfrac{\partial u_1(s_1; s_2^{\text{¤}})}{\partial s_1} > 0$ for all $s_1 < s_1^0$. Lemma 4.2 (4.10) and (4.12) imply that $\text{sign}\left[\dfrac{\partial b_1(s_2^{\text{¤}}; \textcircled{R}_1)}{\partial \textcircled{R}_1}\right] = \text{sign}\left[\dfrac{\partial f_1(\textcircled{R})}{\partial \textcircled{R}_1}\right] = 0$ for all $\textcircled{R}$. Now if $\textcircled{R}_1^{\text{¤}} > 0$ and $\dfrac{\partial f_1(\textcircled{R}^{\text{¤}})}{\partial \textcircled{R}_1} < 0$ then using the facts of the previous two sentences we find that $s_1^{\text{¤}} < s_1^0$ and $\dfrac{\partial u_1(s_1^{\text{¤}}; s_2^{\text{¤}})}{\partial s_1} > 0$. If $\textcircled{R}_1^{\text{¤}} > 0$ and $\dfrac{\partial f_1(\textcircled{R}^{\text{¤}})}{\partial \textcircled{R}_1} > 0$ then by a similar reasoning $s_1^{\text{¤}} > s_1^0$ and $\dfrac{\partial u_1(s_1^{\text{¤}}; s_2^{\text{¤}})}{\partial s_1} < 0$. Thus, we have show if $\textcircled{R}_1^{\text{¤}} > 0$ then $\dfrac{\partial f_1(\textcircled{R}^{\text{¤}})}{\partial \textcircled{R}_1} \dfrac{\partial u_1(s_1^{\text{¤}}; s_2^{\text{¤}})}{\partial s_1} < 0$.

For $\textcircled{R}_1^{\text{¤}} < 0$ we find by similar reasoning that $\dfrac{\partial f_1(\textcircled{R}^{\text{¤}})}{\partial \textcircled{R}_1} \dfrac{\partial u_1(s_1^{\text{¤}}; s_2^{\text{¤}})}{\partial s_1} > 0$.

Lemma 4.2 (4.11) implies that the second part of the right hand side of (A.10) $\dfrac{\partial u_1(s_1^{\text{¤}}; s_2^{\text{¤}})}{\partial s_2} \dfrac{\partial f_2(\textcircled{R}^{\text{¤}})}{\partial \textcircled{R}_1} < 0$ if G is a carrot game for player 1 while $\dfrac{\partial u_1(s_1^{\text{¤}}; s_2^{\text{¤}})}{\partial s_2} \dfrac{\partial f_2(\textcircled{R}^{\text{¤}})}{\partial \textcircled{R}_1} > 0$ if G is a stick game for player 1.

Thus if $\textcircled{R}_1^{\text{¤}} > 0$ and G is a carrot game for player 1, then $\dfrac{\partial u_{p1}(\textcircled{R}_1^{\text{¤}}; \textcircled{R}_2^{\text{¤}}; f_1; f_2)}{\partial \textcircled{R}_1} < 0$. Alternatively, if $\textcircled{R}_1^{\text{¤}} < 0$ and G is a stick game for player 1, then $\dfrac{\partial u_{p1}(\textcircled{R}_1^{\text{¤}}; \textcircled{R}_2^{\text{¤}}; f_1; f_2)}{\partial \textcircled{R}_1} > 0$. But then Principal 1 can increase his utility in each case by changing $\textcircled{R}_1^{\text{¤}}$ marginally in the direction of zero. ¤

Proof of Proposition 4.2: Let G be a carrot game for player 1. Then by Proposition 4.1 the equilibrium incentives $\textcircled{R}^{\text{¤}} = (\textcircled{R}_1^{\text{¤}}; \textcircled{R}_2^{\text{¤}})$ must satisfy $\textcircled{R}_1^{\text{¤}} < 0$.

Let $(s_1^\pi; s_2^\pi)$ be the equilibrium in $G(®^\pi)$, let $(s_1; s_2)$ be the equilibrium in $G(0)$ (the original game), and let $(s_1^0; s_2^0)$ be the equilibrium in $G(®_1^\pi; 0)$ (the game when principal 1 chooses his equilibrium incentives and principal 2 chooses his incentives for the original game).

By Lemma 4.2 (4.10) and (4.12), and $®_1^\pi < 0$ we have that $s_1^0 < s_1$ if and only if $\frac{@u_2}{@s_1} < 0$. Hence, $u_2(s_1; s_2) < u_2(s_1^0; s_2) \cdot u_2(s_1^0; s_2^0) \cdot u_2(s_1^\pi; s_2^\pi)$ where the ⁻rst inequality follows from the monotonic externalities and the second inequality follows from the fact that s_2 is available but s_2^0 is chosen in $G(®_1^\pi; 0)$, and the last inequality follows since $®_2^\pi$ is chosen and $®_2 = 0$ is available..¤

Proof of Proposition 4.3: By Proposition 3.1 it follows that G can be a stick game for both players only if the best response functions of both players have the same slope. If they are downward sloping then the externalities must move in the same direction. If they are upward sloping, then externalities must move in opposite directions.

We show the case where reaction functions are downward sloping and externalities are negative, i.e., $b_1^0(s_2) < 0$, $b_2^0(s_1) < 0$, $\frac{@u_1}{@s_2} < 0$, and $\frac{@u_2}{@s_1} < 0$. The other cases can be shown by the same type of argument.

By Proposition 4.1 we have that the equilibrium $®_1^\pi > 0$ and $®_2^\pi > 0$. Let $(s_1; s_2)$ be the equilibrium of $G(0; 0) = G$, and let $(s_1^\pi; s_2^\pi)$ be the equilibrium of $G(®_1^\pi; ®_2^\pi)$. We show that $s_1^\pi \cdot s_1$ implies $s_2^\pi > s_2$, and $s_2^\pi \cdot s_2$ implies $s_1^\pi > s_1$.

If $s_1^\pi \cdot s_1$, then $s_2^\pi = b_2(s_1^\pi; ®_2^\pi) > b_2(s_1^\pi; 0) , b_2(s_1; 0) = s_2$. The equalities come from de⁻nitions. The strict inequality follows from applying (4.12) for changes in $®_2$ to obtain $\frac{@b_2(s_1; ®_2)}{@®_2} > 0$ and noting that $®_2^\pi > 0$. The weak inequality follows from $b_2^0(s_1) < 0$ and $s_1^\pi \cdot s_1$.

If $s_2^\pi \cdot s_2$, then $s_1^\pi = b_1(s_2^\pi; ®_1^\pi) > b_1(s_2^\pi; 0) , b_1(s_2; 0) = s_1$ by the same type of reasoning given in the previous paragraph.

From this we conclude that either $s_2^\pi > s_2$ or $s_1^\pi > s_1$.

If $s_2^\pi > s_2$, then player 1 is worse o® since $u_1(s_1^\pi; s_2^\pi) < u_1(s_1^\pi; s_2) \cdot u_1(s_1; s_2)$, where the strict inequality follows from $s_2^\pi > s_2$ and $\frac{@u_2}{@s_1} < 0$, and the weak inequality follows from utility maximization of player 1 in $G(0)$.

If $s_1^\pi > s_1$, then player 2 is worse o® by the same type of argument.¤

Proof of Corollary 4.2: Since G is a stick game for each player we have by Proposition 4.3 that at least one player is worse o® in $¡$. Since the players are symmetric it follows that they will be treated symmetrically in a symmetric equilibrium of $¡$ and thus both are worse o® than in G.¤

Proof of Proposition 5.1: Suppose the game is Stackelberg solvable and let $(\nabla_1; \nabla_2)$ be de⁻ned by $(s_1; s_2)$. Conditions (2.1) and (2.2) are not satis⁻ed for player 1 since by (5.1) we cannot ⁻nd a pair $(s_1^0; b_2(s_1^0))$ satisfying $u_1(s_1^0; b_2(s_1^0)) > \nabla_1$. The same argument holds for player 2 by (5.2).¤

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