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The Number Crunch game: A simple vehicle for building algebraic reasoning skills

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Abstract

A newspaper numbers game based on simple arithmetic relationships is discussed. Its potential to give students of elementary algebra practice in semi ad-hoc reasoning and to build general arithmetic reasoning skills is explored.

1 Introduction

In recent months, Australia's national daily newspaper, *The Australian*, has published, several times per week, a puzzle called *Number Crunch*. An example is shown in Figure 1, with the general layout of this class of puzzle depicted in Figure 2. The basic properties and features of this kind of puzzle are discussed in section 3. Over the past months, I have developed an interest in this puzzle, and in the methods one may use for its solution. My principal interest is in the potential of *Number Crunch* for helping my students develop basic skills of middle-school algebra and the mathematical reasoning appropriate for that level. The following sections give the background to my use of this game at Bond University, and describe the manner in which it is presented to the students.

2 The *Elementary Mathematics* unit at Bond

From time to time, I teach a unit at Bond University known as *Elementary Mathematics*. In a nutshell, this is a crash course (twelve weeks) in very basic algebra for beginning undergraduate students who have not mastered it at high-school. These are primarily prospective business students, however some students of other disciplines such as medicine also take the class. In brief, the content is:

1. Introduction to algebra, rearranging simple equations to solve for an unknown; use of simple identities.

2. Linear equations in one unknown.

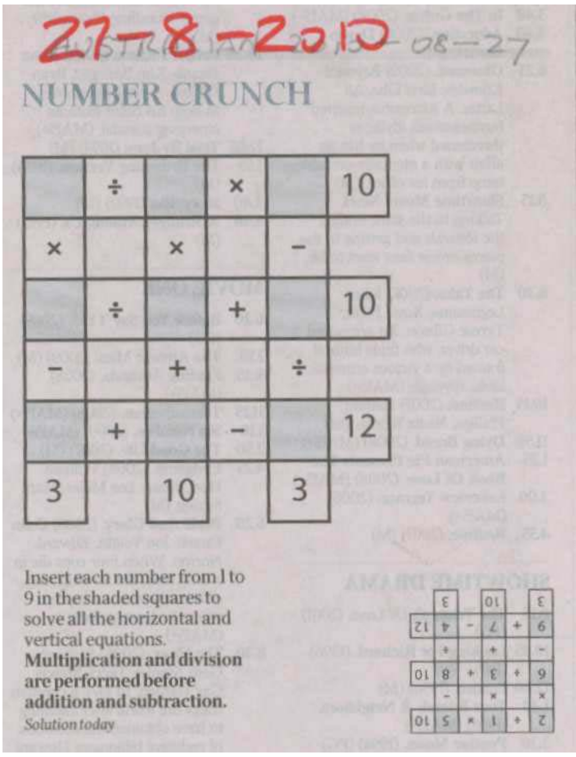

Figure 1: *Number Crunch* from *The Australian* newspaper 27th August 2010.

a	op_1	b	op_2	c	n_1
op_7		op_9		op_{11}	
d	op_3	e	op_4	f	n_2
op_8		op_{10}		op_{12}	
g	op_5	h	op_6	i	n_3
n_4		n_5		n_6	

Figure 2: General structure of *Number Crunch*.

3. Properties of straight lines (slope, intercept); shift of origin.

4. The functions $y = ax^2 + bx + c$, $y = x^{-1}$ and $y = x^k$ for small positive integers k and $k = 1/2,\ -1/2$

5. Index laws & the functions e^x, $\log_e x$ with examples from business and science.

6. Linear inequalities in one and two variables.

7. Very elementary introduction to vectors & matrices, with applications.

8. Word problems.

Students enrolled in this unit typically have great difficulty in mastering what amounts to a condensation of the best part of around three years of high-school algebra. Since they have only twelve weeks in which to come to grips with this material, it is hardly surprising that problems occur. Many or most of these students had opted for the easier "Maths A" path at high-school in the Australian state of Queensland; a course of study containing no algebra whatsoever. Upon arrival at university, they are predictably informed that some level of algebra competence is required to embark on their degree studies. One such unit consists of very basic differential calculus in one and two variables (up to the chain rule), very simple single-variable definite and indefinite integrals, linear programming, and matrices and vectors with applications.

Student difficulties with even basic mathematics at university level are painfully obvious to both the students themselves and those whose responsibility it is to teach them. As well as the glaringly obvious, there are some slightly more subtle issues. Quite distinct from the acquisition of specific content knowledge, students who persevere with "normal" algebra and calculus in senior school (Maths B in Queensland, or its equivalent in other states) develop a familiarity with the basic vocabulary, modes of reasoning, and some of the basic techniques, even if they are not brilliant at mathematics. This acclimatization takes place over years, and students at least have been exposed to "mathematical thinking", jargon and methods, and so some quantity of this material, perhaps small, "sticks". For those poor souls enrolling in *Elementary Mathematics* at Bond, most of whom have *not* been exposed to much mathematics at school, there remains the task of not only learning the material, but also coming to grips with the language, jargon, and modes of thought required even for basic algebra, and even *elementary arithmetic*, in some cases, and all of this in twelve weeks. From the student point of view, it must be worse than learning a whole new, totally unfamiliar language, as even the *basic vocabulary* is unknown. When teaching such a class, one needs to be careful in the use of language, as significant numbers of the students do not know the technical meanings of words such as *term, factor, product, sum, difference, quotient, remainder* and many routinely reach for a calculator when required to multiply, or in some cases, *even add*, single decimal digit numbers. On rereading these claims now, they seem incredible to me, but I know from first-hand experience that they are, regrettably, all too true.

Thus, I am expected to bring these students up to scratch in a mere twelve weeks, where the material covered would have otherwise taken three or four years at high-school. I taught this subject last at Bond in the January 2009 trimester and the failure rate was 40%. Given this milieu, any reasonable means by which the basic arithmetic and algebraic reasoning skills of the students may be strengthened, in a supportive environment, is surely worthy of consideration. I have formed the view that the *Number Crunch* game has potential in this regard. The current plan is to present examples of this puzzle in Bond's January 2011 trimester, and then require the students to solve one or two similar puzzles as an assignment.

In the remainder of the paper, we summarize the general features of the *Number Crunch* game, and then go on to consider some representative examples and the modes of reasoning used to solve them. The intention is then to distil from all of this some general principles and approaches to be recommended to the students.

In using the *Number Crunch* game as a vehicle for the learning and practice of basic algebra, the general aims are, to the extent possible in the short time available, develop or hone the following abilities in the students:

1. casting of a given problem into a standard, prescribed algebraic form;
2. efficient search for tight bounds on unknowns;
3. appreciation of the rôle and power of *reductio ad absurdum;*
4. a sense for when the use of semi *ad-hoc* reasoning may be appropriate;
5. an appreciation of the value of having a "toolkit" of suitable heuristics at hand for a given problem class;
6. modelling and problem-solving skills;
7. an awareness that sometimes, a reasonable way to attack a problem is to eliminate obvious non-solutions, and reduce the problem to a relatively modest number of remaining cases[1], and,
8. skill and experience in the use of everyday technology such as Excel for simple tabulation of such a modest number of cases (possible solutions or partial solutions)[2].

3 Essential features of the *Number Crunch* problem class

As seen from Figure 1, a solution is shown published adjacent to each puzzle, but upside-down. There is no claim that solutions are unique, but this is implicit in that only one

[1] This is reminiscent of the old elephant hunting joke, in which "Mathematicians hunt elephants by going to Africa, throwing out everything that is not an elephant, and catching one of whatever is left." [3]

[2] The potential of spreadsheets to "bridge the gap" from number to algebra and to help build algebraic understanding is well-documented [1], [2], [4], [5].

solution is ever published. To date, I have found just one instance of non-unique solution, and this is discussed further in section 6.1.

With reference to Figure 2, each op_k for $1 \leq k \leq 12$ denotes one of the four fundamental operations of integer arithmetic: addition, subtraction, multiplication and division. In some 50 published instances seen, I have noted that all intermediate results are positive integers, so that no division operation ever produces a non-zero remainder. The n_j for $1 \leq j \leq 6$ denote integers, which in all published instances to date have been non-negative.
The nine shaded cells represent unknowns for the problem and are always a permutation of the integers 1 to 9 inclusive. In recent months, I have seen at least one instance of the puzzle on a website where the solution implies that the traditional order of operations is violated, i.e., addition or subtraction is done before multiplication or division. It seems that the newspaper also initially erred in this fashion. If they can be believed, some Internet blogs revealed that lobbying from an Australian university mathematics department staff member finally convinced the newspaper to conform to centuries of mathematical tradition by directing a computer programmer to alter code so as to generate correct future puzzles. Thus, the line "multiplication and division are performed before addition and subtraction" may be seen in Figure 1. We summarize the essential features of the puzzle:

1. Six simultaneous, non-linear equations in nine integer unknowns.
2. Each unknown occurs in exactly two of the six equations.
3. Each equation contains exactly three unknowns.
4. The solution vector is always a permutation of the integers $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$, and so the cardinality of the search space for a zero-knowledge, brute-force search is therefore $9! = 362\,880$.
5. Operation counts are invariant: we always have three of each of the four arithmetic operators present in the puzzle.
6. Only one solution is ever published, on the same page as each puzzle, but at least one instance of multiple solutions has been found (see section 6.1).

4 Possible solution methods

1. Formulation and solution as an under-determined, constrained nonlinear integer program, e.g., in Excel's Solver. For this *sledgehammer* approach, it is difficult to formulate the constraint that any feasible solution must be a permutation of the integers 1 through 9.
2. Another *sledgehammer* approach is brute force linear search of the solution space by writing and running 3GL code with nine nested forloops, one for each unknown.

This approach is unwieldy, inefficient, clumsy (constraints which are not easy to implement implicitly must be explicitly coded). It also gives beginning algebra students nothing, but may be worthwhile in showing computing students how *not* to solve a problem.

3. Elimination of certain unknowns by algebra. This method involves a lot of manipulation, and has the negative side-effect of making implicit constraints more obscure. It is not considered here.

4. Careful reduction of solution possibilities for strategically-chosen unknowns by algebraic and arithmetic reasoning. This is done until either the solution just drops out, or a very short list of possibilities appears, from which the solution(s) remain after the rest are eliminated. Limited tabulation may then be done, e.g., in Excel. The bulk of the paper is concerned with this approach as it has the potential to build reasoning skills in the students.

5 Classroom approach

How should this problem be treated in the tertiary classroom where the aim is introduce students to basic algebra? It is recommended that the teacher give some general advice and heuristics, plus two or three examples, carefully choosing these to make sure the essential elements of both puzzles and typical solution approach (see sections 6.2 and 6.1) have been covered.

When presenting examples, it makes sense for the teacher to try to strike a balance between a totally prescriptive approach to the solution, and a completely blank page, where students are entirely on their own. Fortunately, *Number Crunch* is the kind of problem where a totally prescriptive approach is not really possible, at least at a level consistent with beginning algebra students. Some general heuristics can be given, and the students warned that they must "put their thinking caps on". From a more advanced algebraic standpoint, to use just algebraic manipulation will, even in the most optimistic of scenarios, reduce the answer to a three-parameter solution vector, which may then generate solution(s) as the parameters run through their permissible values. Such an approach is not considered here, as this is not our intention. Rather, we seek to build reasoning skills in the students and emphasize approach #4: use of algebraic and arithmetic reasoning. We argue that the *Number Crunch* problem seems to be at an ideal level: not too trivial, but not overwhelmingly difficult either. It will typically demand controlled application of the "case-by-case" reasoning that is often required in mathematical argument, often as part of the solution of some larger problem. Careful control of the number of cases is necessary lest this become unmanageable.

5.1 Heuristics

These are "rules of thumb" which are often useful in making progress on a problem, though there is no general guarantee that a solution will be found, or even any progress at all will be made. Several heuristics useful for *Number Crunch* are stated here and illustrated by example.

H1. **Eliminate unit denominators if possible**. If division, e.g. e/f occurs in an equation, check first if $f = 1$ is possible/feasible, by trial substitution in the other equation in which f occurs. If you cannot rule out $f = 1$ by reasoning from one or more of the other equations, then this generally makes the job more difficult. If $f = 1$ is impossible, then $e/f \in \{2, 3, 4\}$. This is so, since $e \leq 9$ and $f \geq 2$ (remember, we ruled out $f = 1$). The published example of 24 September 2010 gave rise to the usual six equations, two of which were $d + e/f = 7$ and $cf/i = 12$. Now if $f = 1$ here, then, we also have $c = 12i$, which is impossible, since $i \geq 1$ and $c \leq 9$. Thus, we conclude that $f \geq 2$. A simple step such as this is a good example for the students to see *reductio ad absurdum* in action. It must be emphasized that this is a very powerful tool for the working mathematician, and is often the best way to establish a result. In my experience, many students are reluctant to use this powerful tool of "proof by contradiction". The *Number Crunch* game is a fruitful source of instances where this technique may be used, and in my view, this alone is reason enough for its use in the development of reasoning skills in students of elementary algebra.

H2. **Prime factorization of certain coefficients**. In the example of section 6.1 is a case where we make good use *Euclid's Lemma*: if a prime p divides ac, then p divides a or p divides c. Equation (7) is $ac = 10b$, and since ac is a multiple of 5 and 5 is prime then, then either a or c is a multiple of 5. There is only one multiple of 5 in the interval 1 to 9, namely 5 itself, so either $a = 5$ or $c = 5$. Note that the "or" in the previous sentence is *exclusive or*. Also, we cannot conclude from $ac = 10b$ that a or c is a multiple of 10, as 10 is not prime; counterexample is $a = 4$, $c = 5$, $b = 2$. For the present problem class, this is impossible anyway as multiples of 10 are out of bounds, but here is a good opportunity for the teacher to mention the importance of prime numbers. Factorization is not so useful when coefficients have more than two prime factors or have repeated prime factors; e.g., $ac = 12b$ allows many more possibilities than $ac = 10b$. The present heuristic coupled with H4 makes for a powerful combination to keep the number of combinatorial possibilities for solutions in check.

H3. **Denominators cannot exceed 4**. Since $a/b \geq 2$ or equivalently, $a \geq 2b$, we must have $b \leq 4$ since $a \leq 9$ and $a \neq b$.

H4. **Only 31 different products are possible**. For equations of the form $ab = kc$ where k is a constant and a, b, c are unknowns, e.g., $ab = 15c$, only certain products are possible. In general, if a and b are unknowns, and nothing except $a \neq b$ is known about them, then there are 31 possible values of ab, as shown in Table 1. If the possible values of a or b have been narrowed down to small sets, then the set of products may, in general, be reduced. For example, in the case $ab = 15c$, we may deduce that $ab \in \{15, 30, 45\}$ and that $c \in \{1, 2, 3\}$.

H5. **Controlled tabulation**. If progress cannot otherwise be made, one may tabulate all feasible[3] values of an unknown or combinations of values of two or more unknowns. Then the equations of the problem may be used to calculate the remaining unknowns. Finally, eliminate those not satisfying the conditions of the problem (the numbers 1 to 9 each occurring just once in any solution). This technique must be used with great caution as it has the potential to generate a huge number of cases. Further, if one is not sufficiently organized and systematic, it is easy to overlook some combinatorial possibilities when constructing a table. The general tabulation approach is illustrated in a small way in section 6.1, and in a much more extensive way in section 6.2.

Table 1: Possible products

2	3	4	5	6	7	8	9
10	12	14	15	16	18	20	21
24	27	28	30	32	35	36	40
42	45	48	54	56	63	72	

6 Examples

We consider two examples, chosen ahead of many possible others for illustration of:

- multiple solutions (section 6.1)
- *reductio ad absurdum*
- semi *ad hoc* reasoning
- the heuristics of section 5.1
- possibly one of the more extreme cases where extensive tabulation seems to be the easiest approach (section 6.2)

6.1 Worked solution for published puzzle of 27th August 2010

This example has two solutions, just one of which was published. Representing the unknowns as $a, b, c, d, e, f, g, h, i$, and transcribing the operations to a form of equations corresponding most directly to that published leads to the following underdetermined set of six simultaneous nonlinear equations in nine unknowns.

[3] In general mathematical work, *feasible* means that all constraints are satisfied. In the present context, the meaning is similar: do not tabulate values or combinations of values which are clearly impossible. Thus, some (non-obvious) infeasible values or combinations may still remain.

$$\begin{aligned} a/b\times c &= 10 \quad (1)\\ d/e+f &= 10 \quad (2)\\ g+h-i &= 12 \quad (3)\\ ad-g &= 3 \quad (4)\\ be+h &= 10 \quad (5)\\ c-f/i &= 3 \quad (6) \end{aligned}$$

We prefer to rewrite each equation without subtraction and, where only multiplication and division are present, without division. For the present set, this results in the following equivalent set of equations (no unknown may be zero, so removal of any possible singularity such as that in eq 1, is a non-issue):

$$\begin{aligned} ac &= 10b \quad (7)\\ d/e+f &= 10 \quad (8)\\ g+h &= i+12 \quad (9)\\ ad &= g+3 \quad (10)\\ be+h &= 10 \quad (11)\\ c &= f/i+3 \quad (12) \end{aligned}$$

Our general approach is to constrain certain unknowns to only a few possible values, hopefully just one value. Various heuristics, as summarized in section 5.1, may often be used to achieve this. If this proves fruitless, we may resort to enumerating a potentially large number of possible cases. This last approach should be used with great caution, as it has a tendency to produce a great deal of work. It also detracts from the main object of the whole exercise, which is to use careful reasoning to implicitly eliminate many cases, without having to consider each one separately. However, once a few values are established without doubt, it is usually a good idea to then construct a simple tabulation of known values, with empty spaces for the unknowns. It then becomes clear which values from 1 to 9 have been "used up", and therefore, cannot be the value of any remaining unknown. Let's begin. Invoking H1 of section 5.1, we see that eq (7) implies that $a = 5$ or $c = 5$. We now consider these in Parts 1 and 2 respectively.

Part 1. $a = 5$. Each of the sub-parts (a) through (i) here assumes $a = 5$.

(a) Eq 10 implies that $5d \in \{4, 5, 6, 7, 8, 9, 10, 11, 12\}$, so $d = 1$ or $d = 2$. But eq 8 shows that $d \neq 1$ (if d were 1 the fraction d/e can only be integer if e is also 1). Thus, we conclude that $d = 2$.

(b) Again, from eq 8, since d/e can only be integer if $e = 1$ or $e = 2$, we conclude that $e = 1$, since d is already 2.

(c) Since we know the values of both d and e, eq 8 immediately yields $f = 8$, and eq 10 gives $g = 7$.

(d) Since we know (assuming $a = 5$) that $g = 7$, eq 9 now reads $h = i + 5$. Since 7 is already "taken" by g and 8 by f, this means that $h \in \{6, 9\}$.

(e) We can also see that h cannot be 9 since eq 11 would then imply that $be = 1$. Even if we didn't already know that $e = 1$, this would still be impossible since b and e cannot both be 1. Thus, we now have $h = 6$.

(f) Since $h = 6$ and $e = 1$, we now see from eq 11 that $b = 4$.

(g) As noted above, it is useful at some point to tabulate known values. These are summarized in Table 2.

Table 2: Partial solution for Part 1, as at step (g)

a	b	c	d	e	f	g	h	i
5	4		2	1	8	7	6	

(h) Eq 9 now yields $i = 1$, which is impossible, because already $e = 1$.

(i) The entire chain of reasoning of Part 1 now shows that the assumption $a = 5$ is false, since valid reasoning has led to a contradiction. This process is known as (dis)proof by contradiction, or *reductio ad absurdum*. As working mathematicians, we know that it is a very powerful proof technique, commonly and widely used in mathematical reasoning and elsewhere. It is a technique with which students should be thoroughly familiar. For some reason, many students are reluctant to use it, yet it is probably the most powerful general technique available.

Part 2. $c = 5$. We have worked pretty hard to establish that $c = 5$, and it is possible that a simpler approach may have been used. Nevertheless, we now proceed with the assurance (no longer an assumption) that $c = 5$. Of course, the "known" values in Table 2 must now be discarded, as they were derived under the assumption that $a = 5$, now known to be false.

(a) With $c = 5$, eq 7 now simplifies to $a = 2b$. Use this to eliminate a in eq 10 to yield $2bd = g + 3$. Now since $1 \leq g \leq 9$, and $2bd$ is even, we must have $2bd \in \{4, 6, 8, 10, 12\}$. This is equivalent to $bd \in \{2, 3, 4, 5, 6\}$. But $c = 5$, and 5 is prime, so $bd \in \{2, 3, 4, 6\}$.

(b) We now eliminate $bd = 2$. Here is another example of (dis)proof by contradiction. Suppose that $bd = 2$. Since 2 is prime, this means that $(b, d) = (1, 2)$ or $(b, d) = (2, 1)$. In the former case, eq 8 then reads $2/e + f = 10$, which only makes sense if $e = 1$. But $b = 1$, so this cannot be. In the latter case, eq 8 reads $1/e + f = 10$, which is also impossible unless $e = 1$, but $d = 1$ already. Thus, $bd = 2$ is impossible, and we are reduced to $bd \in \{3, 4, 6\}$.

(c) Reasoning similar to that of the previous paragraph shows that $bd \neq 3$. We are left with $bd \in \{4, 6\}$.

(d) Since $\{4, 6\}$ is a small set *and* its elements are also small *and* they have few factors, we may tabulate possible values of b and d. We know that $c = 5$, so this can also be inserted in each row. Eq 7 also yields the value of a in each case, since b and c are known. Eq 10 yields g. The result is Table 3.

Table 3: Partial solution for Part 2, as at step (d)

	a	b	c	d	e	f	g	h	i
Case 1	2	1	5	4			5		
Case 2	8	4	5	2			13		
Case 3	2	1	5	6			9		
Case 4	12	6	5	1			9		
Case 5	4	2	5	3			9		
Case 6	6	3	5	2			9		

(e) With reference to Table 3, Cases 1, 2, 4 may be eliminated on the grounds that duplicate or out-of-bounds values appear. We are now left with the possibilities shown in Table 4.

Table 4: Partial solution for Part 2, as at step (e)

	a	b	c	d	e	f	g	h	i
Case 3	2	1	5	6			9		
Case 5	4	2	5	3			9		
Case 6	6	3	5	2			9		

(f) With reference to Table 4, consider Case 5. Eq 12 tells us that $f = 2i$, and $2, 4$ are already "taken", so we must have $f \in \{6, 8\}$. But this means that $i \in \{3, 4\}$ and these are also "taken". This rules out Case 5. We are now down to Cases 3 and 6 as shown in Table 5.

Table 5: Partial solution for Part 2, as at step (f)

	a	b	c	d	e	f	g	h	i
Case 3	2	1	5	6			9		
Case 6	6	3	5	2			9		

(g) With reference to Table 5, in both Cases 3 and 6, the even numbers 2 and 6 are both "taken", so that eq 12 tells us that $i = 4$ and $f = 8$. We now have Table 6.

(h) With reference to Table 6, eq 9 yields $h = 7$ in both Cases 3 and 6. Eq 11 then tells us that $be = 3$, so in Case 3, $e = 3$ and in Case 6, $e = 1$. Our

Table 6: Partial solution for Part 2, as at step (g)

	a	b	c	d	e	f	g	h	i
Case 3	2	1	5	6		8	9		4
Case 6	6	3	5	2		8	9		4

final two solutions are shown in Table 7. It is important to now verify that all equations 7 through 12 are satisfied. This is indeed the case.

Table 7: Final solution for Part 2, as at step (h)

	a	b	c	d	e	f	g	h	i
Case 3	2	1	5	6	3	8	9	7	4
Case 6	6	3	5	2	1	8	9	7	4

6.2 Worked solution for published puzzle of 8th October 2010

Computer scientists use the convenient abbreviations *mulops* and *addops* to refer to multiplication-like operators and addition-like operators respectively when defining a programming language syntax (grammar). Thus, *mulops* are typically multiplication or division and *addops* are usually addition or subtraction. Referring to eqs 13 to 18, we may see the present example is somewhat unusual in that all 6 "mulops" occur in the first three equations, and then all 6 "addops" are in the last three equations. It then becomes difficult to get reductions due to only additive relations in the last three equations. For such problems, and perhaps other similar ones, a useful method is careful selection of low-cardinality sets arising from the mulops, then tabulation of a modest number of possibilities (perhaps a few dozen or so, at most) in a spreadsheet such as Microsoft Excel. For the present example, the steps involved in such an approach are listed in section 6.3.

The equations for this puzzle are:

$$ab = 12c \tag{13}$$
$$df = 15e \tag{14}$$
$$gh = 14i \tag{15}$$
$$a + g = d + 10 \tag{16}$$
$$b + e = h + 6 \tag{17}$$
$$c + f = i + 7 \tag{18}$$

The structure of this puzzle is such that the equations do not lend themselves readily to reasoning as in the previous example. We now choose to apply the method of outlined in H5 of section 5.1. The spreadsheet Excel is used to assist with tabulation and constraint-checking

6.3 Using Excel to tabulate possibilities

We describe an approach suitable for the present example, where it seems that not much progress can be made without enumeration. The aim here is to find the most economical means of enumerating possibilities. Here are the steps for our example.

1. With reference to eq 13, invoking H4 yields $ab \in \{12, 24, 36, 48, 72\}$ and $c \in \{1, 2, 3, 4, 6\}$.

2. With reference to eq 14, invoking H4 yields $df \in \{15, 30, 45\}$ and $1 \le e \le 4$.

3. With reference to eq 15, invoking H4 yields $gh \in \{14, 28, 42\}$ and $1 \le i \le 3$.

4. As noted, because of the distribution of "mulops" and "addops", it is difficult to use the last three equations to make progress. We now opt to enumerate possibilities for some of the variables. The products ab, df, gh can only take on values from the small sets given in the previous three steps. However, all members of the set of possibilities for ab, i.e., $\{12, 24, 36, 48, 72\}$ have several repeated prime factors. For example, $48 = 2^4 \times 3^1$, so that the number of factors of 48 is $(4 + 1) \times (1 + 1) = 10$. Granted, some of the factors exceed 9 so could not possibly be values of a or b, but there is also the added problem of *order*, in that $(a, b) \neq (b, a)$. This simple consideration leads us to avoid explicit enumeration of possibilities for (a, b) and focus on those for (d, f) and (g, h).

5. For (d, f) we have the six possibilities $(3, 5)$, $(5, 3)$, $(5, 6)$, $(6, 5)$, $(5, 9)$ and $(9, 5)$. For (g, h) we have the six possibilities $(2, 7)$, $(7, 2)$, $(4, 7)$, $(7, 4)$, $(6, 7)$ and $(7, 6)$. This implies that for (d, f, g, h) we have 36 possibilities, being the Cartesian product of these two.

6. The spreadsheet Excel lends itself to tabulation and enumeration of these 36 cases, and also to automatic generation of the values for the remaining variables with simple formulas. We give the following steps.

7. On a fresh worksheet, place the headings `a` through `i` in cells `A1` through `I1` respectively. Select the range `A1:I37`, Formulas tab on the ribbon, then Defined Names group, Create from selection. This step allows us to express our equations in terms of familiar variable names instead of obscure references such as `A2` or `B3`.

8. Then place the pairs $(2, 7)$, $(7, 2)$, $(4, 7)$, $(7, 4)$, $(6, 7)$ and $(7, 6)$ into cells `G2:H7`. Your model should look like Figure 3.

9. For larger Cartesian products, there are more efficient ways in Excel, but as our product set is of modest size, we use a semi-ad hoc approach. The block of cells `G2:H7` must be duplicated five times directly below itself. To make the rest of the process as painless as possible, it is recommended to first select the range `G2:H7` and then place a thick border around it. This may be done by clicking on the border tool button in the Font area of the Home tab on Excel's ribbon (we are

a	b	c	d	e	f	g	h	i
						2	7	
						7	2	
						4	7	
						7	4	
						6	7	
						7	6	

Figure 3: An initial tabulation in Excel.

using Excel 2007). Now that this block has a border we can select it and copy it directly below itself five times.

10. Now we insert the values for d and f. For the first "block" of (g, h) values, we insert the first ordered pair for (d, f), i.e., $(3, 5)$; for the second block, the second ordered pair $(5, 3)$, and so on on until the final pair $(9, 5)$. The bold borders are now useful.

11. Compute e from eq 14. In cell `E2`, place the formula `=d*f/15`. Fill down to `E37`.

12. Compute i from eq 15. In cell `I2`, place the formula `=g*h/14`. Fill down to `I37`.

13. Compute a from eq 16. In cell `A2`, place the formula `=d+10-g`. Fill down to `A37`.

14. Compute b from eq 17. In cell `B2`, place the formula `=h+6-e`. Fill down to `B37`.

15. Compute c from eq 17. In cell `C2`, place the formula `=i+7-f`. Fill down to `C37`.

16. The range `A1:I37` (including headings) contains all possible solutions, and many non-solutions. It is possible without undue effort, to get Excel to check each "solution" (row) for feasibility. The term feasibility means "satisfying all constraints", viz., all values are 1 through 9, and no value is duplicated. Our steps:

    (a) Place the values 1 through 9 in the range `J1:R1`. We now count how many of each of the values appear in each possible "solution".

    (b) Place the formula `=COUNTIF($A2:$I2,J$1)` in `J2`. Drag `J2`'s fill-handle across to `R2`. While the range `J2:R2` is still selected, double-click `R2`'s fill-handle to propagate this formula downwards. We are using a subtlety of spreadsheeting known as a *partially absolute reference*. It is required so that our reference to the first row remains intact when we fill down (this was done by the double-click).

    (c) The values (frequencies) appearing after the previous step indicate how often each value appears in a "solution". A row corresponds to a solution if and only if all these frequencies are 1. Since all frequencies are nonnegative integers, the simplest way to detect this in Excel to place the formula `=PRODUCT(J2:R2)` in `S2` and fill-down (double-click `S2`'s fill handle).

(d) While the range `S2:S37` is still selected, click Conditional Formatting on the home tab of the ribbon, and highlight cells equal to 1. This reveals just one solution:

$$(a, b, c, d, e, f, g, h, i) = (8, 9, 6, 5, 1, 3, 7, 4, 2)$$

7 Summary of student assignment

An assignment, based on *Number Crunch*, is to be given to the students of *Elementary Mathematics* at Bond in the January trimester 2011. This will begin by giving an example of the problem, with its model solution using the last method of section 4. The heuristics of section 5.1 will also be given. Students are then required to work in pairs to solve another two instances of the problem and present a report with the following components:

1. Algebraic formulation of the problem in a prescribed format (required).
2. Use of established valid reasoning steps and perhaps some algebra to reduce the problem to a "small" set of possibilities (which may be a set of just one possibility) (required).
3. Tabulation of a small set of possibilities, and then any impossibilities eliminated, perhaps in Excel (permitted, but not required).
4. The steps of reasoning used throughout to be carefully described (required).
5. All solutions to be found; this implies that if there is only one solution, valid reasoning must be given to prove its uniqueness (required).
6. Validation of each solution by substitution into the original problem.

8 Conclusion

A description of the newspaper *Number Crunch* game has been given, with two examples illustrating a range of techniques which may be used for its solution. The author believes it has good potential for beginning and intermediate algebra students for building and strengthening arithmetic and algebraic reasoning skills. Two examples are presented here to give readers a feeling for the kind of reasoning skills that students are expected to develop. The first example lends itself to a sequence of logical arguments to implicitly eliminate non-solutions, while the second is more of an instance of systematic checking of possible cases. Some general heuristics are also discussed, but the general problem structure of *Number Crunch* also lends itself to ad-hoc reasoning, and the hope is that students will come up with some of their own valid reasoning steps. The general *reductio ad absurdum* or "proof by contradiction" is emphasized throughout and it is expected that students will also make use of this very powerful technique. The material will be presented to the students in the form of some worked examples in class in Bond's

January 2011 trimester. The students are then expected to complete an assignment in which they solve one or two puzzles of the same kind and then write a report to justify their solutions. A follow-up paper is planned to present and discuss their performances on this task.

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