

7-22-2017

Exploring the Fundamental Theorem of Arithmetic in Excel 2007

Stephen Sugden

Bond University, ssugden@bond.edu.au

David Miller

West Virginia University, millerd@math.wvu.edu

Follow this and additional works at: <http://epublications.bond.edu.au/ejsie>



This work is licensed under a [Creative Commons Attribution-Noncommercial-No Derivative Works 4.0 License](https://creativecommons.org/licenses/by-nc-nd/4.0/).

Recommended Citation

Sugden, Stephen and Miller, David (2017) Exploring the Fundamental Theorem of Arithmetic in Excel 2007, *Spreadsheets in Education (eJSiE)*: Vol. 4: Iss. 2, Article 2.

Available at: <http://epublications.bond.edu.au/ejsie/vol4/iss2/2>

This In the Classroom Article is brought to you by the Bond Business School at ePublications@bond. It has been accepted for inclusion in Spreadsheets in Education (eJSiE) by an authorized administrator of ePublications@bond. For more information, please contact [Bond University's Repository Coordinator](#).

Exploring the Fundamental Theorem of Arithmetic in Excel 2007

Abstract

This paper discusses how fundamentals of number theory, such as unique prime factorization and greatest common divisor can be made accessible to secondary school students through spreadsheets. In addition, the three basic multiplicative functions of number theory are defined and illustrated through a spreadsheet environment. Primes are defined simply as those natural numbers with just two divisors. One focus of the paper is to show the ease with which spreadsheets can be used to introduce students to some basics of elementary number theory. Complete instructions are given to build a spreadsheet to enable the user to input a positive integer, either with a slider or manually, and see the prime decomposition. The spreadsheet environment allows students to observe patterns, gain structural insight, form and test conjectures, and solve problems in elementary number theory.

Keywords

prime numbers, factorization, spreadsheet

Distribution License



This work is licensed under a [Creative Commons Attribution-Noncommercial-No Derivative Works 4.0 License](https://creativecommons.org/licenses/by-nc-nd/4.0/).

Exploring the Fundamental Theorem of Arithmetic in Excel 2007

Steve Sugden
Bond University, Australia
ssugden@bond.edu.au

David Miller
West Virginia University, USA
millerd@math.wvu.edu

December 2010

Abstract

This paper discusses how fundamentals of number theory, such as unique prime factorization and greatest common divisor can be made accessible to secondary school students through spreadsheets. In addition, the three basic multiplicative functions of number theory are defined and illustrated through a spreadsheet environment. Primes are defined simply as those natural numbers with just two divisors. One focus of the paper is to show the ease with which spreadsheets can be used to introduce students to some basics of elementary number theory. Complete instructions are given to build a spreadsheet to enable the user to input a positive integer, either with a slider or manually, and see the prime decomposition. The spreadsheet environment allows students to observe patterns, gain structural insight, form and test conjectures, and solve problems in elementary number theory.

Contents

1	Introduction	3
2	Number theory and spreadsheets	4
3	Why spreadsheets?	4
4	A brief review of elementary number theory	6
4.1	Factors	6
4.2	Prime numbers	6
4.3	Some basic functions of number theory	8
4.3.1	Simple Excel model for factorization	8
4.3.2	Steps to build the model	8
4.4	The number of divisors function $\tau(n)$	11
4.4.1	Properties of the divisor function $\tau(n)$	12
4.5	The greatest common divisor function $\gcd(m, n)$	13
4.6	The Euler totient function $\varphi(n)$	13
4.7	The Möbius function $\mu(n)$	15
5	Spreadsheet techniques for illustrating the basic functions	16
5.1	Recursive implementation in Excel	16
6	Advanced Excel model to generate prime factorization	16
7	Acknowledgement	21
8	Exercises	21
9	Solutions to exercises	23
10	Conclusion	25

List of Figures

1	Simple Excel model for factorization	9
2	Advanced Excel model for factorization.	21

Submitted September 2010, revised and accepted December 2010.

Key words: prime numbers, factorization, spreadsheet.

1 Introduction

Seventy years ago, number theorist GH Hardy famously asserted "I have never done anything useful"[18]. Once regarded as the purest of pure mathematics, with little or no apparent "application", number theory has in recent decades assumed a role of great importance, due primarily to its direct application to cryptology and cryptography. The practical utility of number theory nowadays is undeniable; its application to mathematical cryptography and cryptology alone being enough to stay the cries of the pragmatists. Although containing some very challenging unsolved problems, it is our belief that it is possible to make the basics of this fascinating branch of mathematics accessible to students at upper secondary level. Since 21st century students are typically well-acquainted with technology, this task has recently been made considerably easier due to two main factors:

1. Easier motivation of student interest due to the immediate application of elementary number theory to cryptography and hence to web ecommerce security processes such as public-key encryption and associated creation, application, and management of keys and digital signatures.
2. The availability of software tools¹, which, with only modest effort, allow illustration of some of the fundamentals of number theory and in particular, concepts and functions which are directly applicable to cryptography.

On the downside, we also find that a significant proportion of 21st century students entering semi-numerate disciplines such as Information Technology at upper secondary and tertiary levels are severely challenged by basic algebra. In the authors' experience, the vast majority of entering tertiary students studying discrete math (CS/IT and other majors), have not covered modular arithmetic or concepts of elementary number theory at school. Thus, for example, they are typically unfamiliar with precise definitions of primality, relative primality, nor with the definition, properties of, or algorithms to compute *greatest common divisor* nor with the *Fundamental Theorem of Arithmetic (FTA)*².

The paper is intended to give a brief review of the very basics of number theory, and then to show how these may be illustrated in the spreadsheet environment. The spreadsheet models accompanying the paper may be used directly in the classroom as tools to investigate properties of prime numbers. Alternatively, teachers may choose to have their students build the models first, and complete steps are provided for this. Thus, the paper may be used in at least two quite different modes, and is targeted to both students and teachers.

¹It is recognized that much more powerful, but specialist packages such as Mathematica, Maple are available. Our choice of Excel is based on its ubiquity and well-known interface, the well-documented willingness of students to embrace it, and clear research evidence that it assists with learning of mathematical concepts. See also our comments in section 3.

²All integers after 1 are uniquely expressible, up to order of factors, as products of primes. A prime itself is regarded as a trivial product consisting of a single factor.

2 Number theory and spreadsheets

Cairns [8, p24] points out that "despite their undeniable importance, prime numbers are largely absent from school curricula". He also points out one of the reasons number theory is absent from school curricula: most material uses modular (or clock) arithmetic. The present work avoids the use of modular arithmetic for this reason³. We would rather concentrate on using spreadsheets to show the beauty of prime numbers to make it more accessible to students. We believe that this field of study may be made very interesting and practical for at least two clear reasons: its immediate application to the very important subject of cryptography, and also the ease with which the basics may be illustrated and investigated in Excel. The field of number theory can be rather abstract and sophisticated. However, there are many essential bedrock concepts, such as the discovery of primes and their elementary properties, including their central role in the structure of the integers under multiplication. The pinnacle or goal of such studies at upper secondary level may be regarded as a thorough understanding of the Fundamental Theorem of Arithmetic (also called the unique factorization theorem). We argue that with proper treatment and enlisting the aid of Excel, such concepts, and those of basic number-theoretic functions, may be made accessible to upper-secondary school students. Furthermore, elementary number theory provides an avenue for students to experience the discovery of new (to them) mathematics [1], [2]. Add this to the interesting history of number theory [14] and, in particular, the Fundamental Theorem of Arithmetic [5], and it is reasonable to expect that students' natural intellectual curiosity will be piqued.

Devlin [9] emphasizes that in a world where our very national livelihood depends on staying ahead of the technology curve, it is crucial that we equip our students with the kind of thinking skills today's world requires. Being able to use computers is one of those skills. We believe that one way to do this is by introducing a variety of mathematics topics, at different levels depending on the population of students, that can be used to motivate and set a foundation for further studies. We agree with Cairns [8] in that we are not advocating the removal of material from the curriculum. However, we are saying that through technology, there is material that is normally not emphasized or not covered in any depth, which can serve to capture the interest of students and spur them on to pursue their interest (or even change it) in Science, Technology, Engineering, and Mathematics (STEM) disciplines.

3 Why spreadsheets?

Why aren't we recommending Mathematica, MATLAB or Maple? These are all fine systems, and very useful, each in its own way and each for its domain of application. Also, each of these products (and others too numerous to mention) has a particular target user group for which it may be the tool of choice.

We are not concerned with the powerful research tools of mathematical researchers, but with software tools suitable for mathematics education. We wish to enlist the support

³For spreadsheet treatments of modular arithmetic, see, for example, [3], [21].

of such tools, beginning with the early grades of junior school. Our view is that the inculcation of the wonder, beauty and power of mathematics must begin as early as possible in the mathematical education of children. What significant and friendly item of mathematical software is on every PC, including home PCs? The answer is obvious: it is the modern, graphical spreadsheet program, of which the prime exemplar is Microsoft Excel.

Our choice of the modern graphical spreadsheet over more powerful, professional tools such as Mathematica, has been made for the following reasons:

1. Utility. The spreadsheet is a general-purpose tool with which we may reasonably expect that almost all upper secondary and tertiary students are nowadays at least somewhat familiar⁴. All is not rosy however, and Lim has recently written about the situation in Australia [16].
2. Ubiquity. Excel, or perhaps its open source or other equivalent, may be expected to be found on essentially every desktop, laptop, netbook and teaching laboratory machine.
3. User-friendliness. Spreadsheets are quite forgiving when an error is made. They don't just fall in a heap and die, but usually give an (admittedly terse) indication of the error.
4. Updated values are (usually) instantly displayed when a quantity in a model is changed. Such rapid feedback ([13] describes it as immediate feedback) allows for very efficient testing of hypotheses in many instances.
5. Useful, transferable skills are acquired by modelers—Excel is now a standard business and scientific tool. Although spreadsheets were created to fulfil a need in accounting and related applications, they now find much broader application. In particular, their application nowadays to assist in the teaching of algebra is beyond any reasonable doubt [22]. It is evident that there is an increasing groundswell of opinion in this regard from mathematics educators and also those who are preparing pre-service mathematics schoolteachers [3], [6], [10], [19], [15], [20], [24], [25].

Further, Calder [11] notes the following additional benefits of spreadsheets in building mathematical understanding: their interactive nature and propensity for linking concepts [7], ability to manipulate large amounts of data [13], and propensity to foster an investigative approach [20].

⁴We must quickly qualify this sweeping statement. Our experience is that, while most students have some spreadsheet experience, for almost all of them, this experience is very shallow indeed. We find that the skills necessary to construct mathematical models in Excel are, almost always, simply not in evidence.

4 A brief review of elementary number theory

In an invited lecture to ICME-7, van Lint [23], addressing the question “What is discrete mathematics and how should we teach it?” mentioned elementary number theory as one of the most important topics in the study of discrete structures at the tertiary level. As we noted in the introduction, number theory, or at least the more elementary aspects of it, has nowadays assumed a great importance, primarily due to its role in Internet cryptography. In the following sections, we show how the fundamentals may be clearly illustrated in the spreadsheet environment, with plenty of scope for students to experiment and test conjectures. We begin with some essential definitions and then go on to look at the basic functions and FTA.

4.1 Factors

A *factor* is a number which divides evenly into a given number. For example, 6 is a factor of 24, since $24 = 6 \times 4$. More generally, we have the

Definition 1 *Let n, d be integers with $d \neq 0$. We say that d is a factor or divisor of n if and only if there exists an integer k such that $n = dk$. We also say that n is a multiple of d , or that d is a factor or divisor of n .*

Example 2 *We may say that 4 is a factor (divisor) of 24 since there is an integer, 6, such that $24 = 4 \times 6$. Alternatively, we could say that 24 is a multiple of 4.*

4.2 Prime numbers

Some common definitions of a prime number tend to be rather clumsy. For example, "Any natural number greater than 1 divisible only by itself and 1" uses many words, and it is relatively short, in comparison to some others found on the web and in modern books. We seek an alternative one. It needs to be a definition which captures the fact that primes have only trivial factors, but which also excludes the number 1. It turns out that all sorts of troublesome exceptions arise if we allow 1 to be prime; it is simpler just to exclude it. In particular, the *fundamental theorem of arithmetic* (see below) becomes invalid. To rephrase our definition, we use a function defined in a later section. In fact we consider several important and fundamental functions of number theory. At present, we shall operate with the “a number greater than one whose only factors are itself and 1” definition.

Example 3 *The integer 29 is prime since the only factors of 29 are 1 and 29. However, the number 28 is not prime since there exists a factor (for example 4) different from 1 and 28. We call 28 composite since it is greater than 1 and not prime.*

Conjecture 4 *(Goldbach, 1742). Every even number greater than 2 is the sum of two primes.⁵*

⁵<http://www.utm.edu/research/primes/glossary/GoldbachConjecture.html>

Remark 5 *At the time of writing, it is unknown whether Goldbach's conjecture is true or not. A \$1,000,000 prize has even been offered for a proof of it⁶. An interesting spreadsheet exercise for students is to look at Goldbach's comet. Details may be found in [4].*

To motivate the next theorem (*Fundamental Theorem of Arithmetic (FTA)*) and to introduce students to the notation in it, we present an example of decomposing an integer into its prime factorization. After stating the theorem, another example is shown to indicate how the general notation applies.

Example 6 *Given the number 148 we use an algorithm (described in more detail below) to search for factors that are prime. We notice that the prime 2 is a factor and $148 = 2 \cdot 74$. Now we search for another prime factor of 74 and find that 2 is a factor again. So $74 = 2 \times 37$ and $148 = 2 \times 2 \times 37 = 2^2 \times 37$. Since 37 is a prime number, we have factored 148 into a product of its prime factors. This is known as the prime factorization of 148, and it is unique.*

Remark 7 *Every positive integer can be written as a product of powers of primes as shown by the next theorem. The way to establish this theorem would be to show that a search using the algorithm (see below) must eventually terminate since the positive integer is finite.*

Theorem 8 (*The unique factorization theorem*) *This theorem is also called the fundamental theorem of arithmetic (FTA), and it has important application to cryptography. Let $n \geq 2$ be an integer. Then there exists a positive integer k and positive integers $a_1, a_2, a_3, \dots, a_k$ and primes p_1, p_2, \dots, p_k such that*

$$n = p_1^{a_1} p_2^{a_2} p_3^{a_3} \dots p_k^{a_k}$$

and this factorization is unique except for the order of the primes p_i . We normally order these such that they form an increasing sequence $p_1 < p_2 < p_3 < \dots < p_k$. For a proof of this basic result, see, for example, Hardy & Wright [17]. FTA proofs are also often presented in modern texts on elementary discrete mathematics, and finally, reasonable treatments may be found online; e.g., at the time of writing, [26].

Example 9 *The number 47 is prime so the prime factorization is trivially represented as 47.*

Example 10 *The number 396 has a prime factorization of $396 = 2^2 \cdot 3^2 \cdot 11$. Comparing the factorization of 396 to the definition we identify $p_1 = 2$, $p_2 = 3$, $p_3 = 11$, $a_1 = 2$, $a_2 = 2$, and $a_3 = 1$.*

⁶<http://www.times-archive.co.uk/news/pages/tim/2000/03/16/timfeafea02004.html>

4.3 Some basic functions of number theory

How do we determine the factors of a positive integer n ? This amounts essentially to a search. For example, to find the factors of 150, we could try all numbers from 1 to 150. If x is a potential factor, we check if $150 \bmod x = 0$. If so, then x is a factor of 150. At first glance, it may appear that we need to check all numbers from 1 to 150 as possible factors of 150. With just a little thought, we see that we can do much better than this. We first observe that factors come in pairs. For example, if 3 is a factor of 150, then there must be another, given by $150/3 = 50$. Second, both of these factors cannot be greater than the square root of 150. Why? Suppose the factors are a and b and that $a > \sqrt{150}$ and $b > \sqrt{150}$. If we multiply these two inequalities together, we get $ab > 150$. This is a contradiction, because $ab = 150$. So, when searching for factors, we only need to search from 2 up to $\sqrt{150}$. Even better, since $\sqrt{150}$ is not an integer, its floor will do, i.e., $\lfloor \sqrt{150} \rfloor = 12$. Once a factor of n is found, say f , its mate is given by n/f .

Example 11 *Following the above algorithm, we determine which integers from 2 to 12 are factors of 150. We find that 2 is a factor and its mate is 75. Continuing we find that 3 is a factor and its mate is 50. Continuing with this algorithm, we find that 5, 6, and 10, are factors and their mates are 30, 25, and 15, respectively. Of course 150 is also a factor and its mate is 1. Therefore the factors are 1, 2, 3, 5, 6, 10, 15, 25, 30, 50, 75, and 150.*

4.3.1 Simple Excel model for factorization

We describe a basic Excel model for determining all the factors of a positive integer, n , from first principles, i.e., the model does not make use of the prime factorization of n ⁷. An important point is one mentioned just above: a search for factors of n need only proceed from 1 up to $\lfloor \sqrt{n} \rfloor$. This saves an enormous amount of unnecessary space and computation in the Excel model. Students may be asked to reflect on this fact: a simple observation or awareness of a basic mathematical principle is frequently enough to save us a lot of time and effort in building models, to say nothing of saving the embarrassment when someone points out a trivial (but in hindsight obvious) change to render our model much simpler and more efficient. Such instances abound in spreadsheet modelling.

4.3.2 Steps to build the model

We describe the steps for building a very simple factorization model in Excel 2007. A snapshot is shown in Figure 1.

1. Open a fresh Excel workbook and place the headings **n** and **limit** in A1 and B1 respectively. In D1 through F1 place the headings **x**, **n mod x**, and **mate** respectively. Make all of these headings centred and bold.

⁷An alternative approach, which *does* make use of FTA and prime factorization, is outlined in section 4.4.1.

2. Select A1:B2 and click on the formulas tab of the ribbon. Then click on Create from Selection in the Defined Names panel of the ribbon. A dialog will appear; just click OK. Save your work.
3. We're going to build a model to find all factors of n , where $1 \leq n \leq 1000$. We'll use a slider (scrollbar) to make changing the value of n easy. Click on the Developer tab of the ribbon, then Insert a scrollbar from the Forms controls. Drag over cells A3:B3 to create the scrollbar. Right-click on the scrollbar and choose Format Control. In the Cell link box, type n . In the minimum value box, type 1 and in the maximum value box, type 1000. Click OK. Play with the slider now and check that it generates values of n in cell A2.
4. The upper limit for our search is $\lfloor \sqrt{n} \rfloor$. Put the formula `=INT(SQRT(n))` in B2.
5. Fill the range D2:D33 with the numbers 1 to 31. Select this range and type x in the Name Box, followed by the Enter key.

n	limit		x	n mod x	mate
150	12		1	0	150
			2	0	75
			3	0	50
			4	2	
			5	0	30
			6	0	25
			7	3	
			8	6	
			9	6	
			10	0	15
			11	7	
			12	6	

Figure 1: Simple Excel model for factorization

6. In E2 type the formula `=MOD(n,x)`. Double-click the fill-handle of E2 to propagate this formula down the column to E32. Now we have an indication of factors of n . These are the values of x in column D) for which $n \bmod x = 0$ (column E).
7. Factors occur in pairs. When we find one factor, x , of n , its mate is given by n/x . This will normally be different from the original factor x . However if n is a perfect square and x is its square root, the second factor is also x , because $n/x = x$. If this happens, it will always be the last on our list, since we only search up to \sqrt{n} . So, in F2, put the formula `=IF(E2=0,n/x,"")`. This formula computes the other factor if the corresponding value of x is a factor of n , and nothing (empty string) otherwise. Double-click to fill down column F parallel to columns D and E. Save your work.

8. To make the model more presentable, with a minimum of distracting unnecessary values displayed, it is desirable to hide all of the contents of columns D through F beyond the point where $x = \lfloor \sqrt{n} \rfloor$. The powerful conditional formatting feature of Excel may be used for this purpose. First select cell D2, then click on Conditional Formatting on the Styles panel of the Home tab of the ribbon. Select New Rule, then *Use a formula to determine which cells to format*. In the edit box, type formula `=D2>limit`. This condition uses a partially-absolute reference to the cell D2 which contains the first value of x . The reason for the form `D2` is to make the reference *column-absolute* but *row-relative*.
9. Click on the Format... button. Select the Font tab and then the Color: drop-down and choose white font. Close the conditional formatting dialogs by clicking OK, then OK again.
10. We now have a prototype of the format we need in cell D2. Use the Format Painter to apply this format to D2:F33. You will see the only relevant part of the table, up to $x = \lfloor \sqrt{n} \rfloor$. This will change as you vary n by using the scrollbar control.
11. Another nice touch is to highlight the factor pairs. This also uses the conditional formatting feature of Excel. First select cell D2, then click on Conditional Formatting again. Select New Rule, then *Use a formula to determine which cells to format*. In the edit box, type formula `=E2=0`. This condition uses a partially-absolute reference to the cell E2 which contains the first value of $n \bmod x$. Again, the reason for the form `E2` is to make the reference *column-absolute* but *row-relative*.
12. Click on the Format... button. Select the Fill tab and then the Color: drop-down and choose a light pastel shade such as light blue. Close the conditional formatting dialogs by clicking OK, then OK again.
13. We now have a prototype of the format we need in cell D2. Use the Format Painter to apply this format to D2:F33.
14. Moving the slider to change the value of n shows us the factor pairs. There may also be highlights due to the most recent conditional formatting rule occurring beyond $x = \lfloor \sqrt{n} \rfloor$. We can remove these by taking advantage of an improvement in the conditional formatting capability in Excel 2007. Simply click on the conditional formatting button again and select Manage Rules Make sure you have chosen the option *Show formatting rules for This Worksheet* from the drop-down list of options. Then, make sure the factor highlighting rule is the second of the two rule (you can move the rules by clicking on the arrows), and finally, simply check the box which says Stop If True for the first rule.
15. If desired, hide column E by right-clicking on the column heading and selecting Hide. This seems to have the unfortunate side-effect of applying the conditional

formatting to column G. Get rid of this by clicking on column heading H, then clicking on the format Painter, then clicking on the column G heading⁸.

4.4 The number of divisors function $\tau(n)$

The model of the previous section is useful for rapidly displaying all factors of numbers up to 1000. In one of the exercises of section 8, we ask you to extend the model to *count* the number of factors. One of the fundamental functions of number theory is a function which does just that: it counts the factors of a positive integer.

Definition 12 *The tau function, $\tau(n)$ is defined for all positive integers n , and is the number of divisors (factors) of n .*

Example 13 *Using the list of factors for 150 of Example 11, we find that $\tau(150) = 12$.*

Example 14 *We have that $\tau(24) = 8$ since there are eight divisors of 24, namely 1, 2, 3, 4, 6, 8, 12, and 24.*

Definition 15 *(Prime number) A positive integer, n , is said to be prime if and only if $\tau(n) = 2$. Non-primes other than 1 are said to be **composite**. The number 1 is neither prime nor composite and is called the unit. Thus, we may speak of a partition of the positive integers into three mutually exclusive classes:*

1. the unit: 1, for which $\tau = 1$
2. the primes: 2, 3, 5, 7, 11, 13, 17, 19, \dots , for which $\tau = 2$
3. the composites: 4, 6, 8, 9, 10, 12, 14, 15, 16, 18, \dots , for which $\tau > 2$.

Suppose n is a positive integer. Then n is a member of exactly one of these classes. See Table 1 for the first 12 positive integers.

Remark 16 *Notice that 1 fails to be prime by our definition, and is excluded quite naturally. Compare this with the somewhat clumsy definition given earlier.*

Remark 17 *Students are used to seeing partitions of the integers. For example, the even and odd integers partition the integers into two sets or 0, positive integers, and negative integers partition the integers into 3 sets. The function τ partitions the positive integers into three sets.*

⁸Our model was created in Excel 2007, however a reviewer has pointed out that this problem does not exist in Excel 2010.

Table 1: The tau function

n	List of divisors	$\tau(n)$	unit	prime	composite
1	1	1	✓		
2	1, 2	2		✓	
3	1, 3	2		✓	
4	1, 2, 4	3			✓
5	1, 5	2		✓	
6	1, 2, 3, 6	4			✓
7	1, 7	2		✓	
8	1, 2, 4, 8	4			✓
9	1, 3, 9	3			✓
10	1, 2, 5, 10	4			✓
11	1, 11	2		✓	
12	1, 2, 3, 4, 6, 12	6			✓

4.4.1 Properties of the divisor function $\tau(n)$

For the divisor function $\tau(n)$, we have the following basic results.

1. $\tau(p) = 2$ if and only if p is prime. This is true as it is simply the definition of prime.
2. $\tau(p^n) = n + 1$ if p is prime. This may be seen from a simple example. Consider the prime factorization $81 = 3^4$. Factors of 81 may be generated by multiplying together 0, 1, 2, 3 or 4 copies of 3 together, generating 1, 3, 9, 27, 81. The cases of 0 and 1 here are somewhat special: 0 copies of 3 means 3^0 , and 1 copy of 3 means 3^1 or just 3 (no multiplication is required in either case). Thus, the total number of factors is $4 + 1 = 5$.
3. If the prime factorization of n is $p_1^{a_1} p_2^{a_2} p_3^{a_3} \dots p_k^{a_k}$, then

$$\tau(n) = \prod_{j=1}^k (a_j + 1) = (a_1 + 1)(a_2 + 1)(a_3 + 1) \dots (a_k + 1) \tag{1}$$

This result extends the previous one to the general case where n is the product of powers of more than one prime. Example: $100 = 2^2 5^2$, so we may choose from 0 to 2 twos and from 0 to 2 fives. We have three choices for each, so the number of factors is $3 \times 3 = 9$. To explicitly generate all of these 9 factors, a number of methods could be used. One possibility is a factor tree; another is a generalized truth-table (one which allows for each column or digit to have a different radix). An Excel model for the latter approach has been constructed by the authors and is available on request.

Example 18 $\tau(2^9) = 9 + 1 = 10$.

Example 19 Consider $n = 84 = 2^2 3^1 7^1$. Therefore, $\tau(84) = (2 + 1)(1 + 1)(1 + 1) = 12$

4.5 The greatest common divisor function $\gcd(m, n)$

Definition 20 *The greatest common divisor function $\gcd(m, n)$. Suppose m and n are natural numbers, not both zero. Then we define their greatest common divisor as the largest integer which is a factor (divisor) of both m and n .*

Example 21 $\gcd(24, 32) = 8$, since 8 is a factor of both 24 and 32, but no larger number is.

Algorithm 22 *How might we compute $\gcd(m, n)$? One way is directly from its definition. We could list all the factors of m and n in two separate lists. Then we could look for the largest number which appears on both lists.*

Example 23 Consider $\gcd(320, 128)$. By the method just mentioned, the factors of 320 are 1, 2, 4, 5, 8, 10, 16, 20, 32, 40, 64, 80, 160, 320, and the factors of 128 are 1, 2, 4, 8, 16, 32, 64, 128. The largest number common to both of these lists is 64; therefore $\gcd(320, 128) = 64$.

Example 24 $\gcd(10^{12}, 12^{10}) = 2^{12} = 4096$. Can you see why?

Definition 25 *Two integers m, n are said to be relatively prime or coprime, if $\gcd(m, n) = 1$.*

Example 26 Since $\gcd(12, 11) = 1$, the numbers 12 and 11 are coprime. However, 12 and 8 are not coprime, since $\gcd(12, 8) = 4$.

4.6 The Euler totient function $\varphi(n)$

This function is important for RSA public-key cryptography—a widely-used data security system for Internet ecommerce. Spreadsheet and related materials for RSA, suitable for upper-secondary and early tertiary students, are available from the authors.

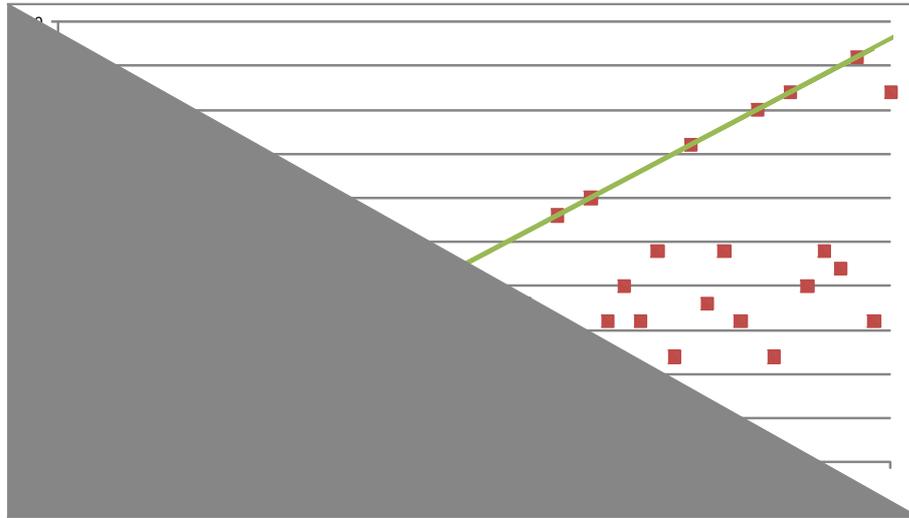
Definition 27 *The Euler totient function, $\varphi(n)$ is defined to be the number of integers, m , in the range $1 \leq m \leq n$, that are coprime to n .*

Example 28 Consider $\varphi(12)$. Of the positive integers not exceeding 12, only the numbers 1, 5, 7, 11 are coprime with 12; thus, $\varphi(12) = 4$.

Example 29 Consider $\varphi(11)$. Of the positive integers not exceeding 11, all except 11 itself are coprime with 11; thus $\varphi(11) = 10$.

Example 30 *There is another way to compute the totient of a number, provided we know its prime factorization. We use the formula*

$$\varphi(n) = n \prod_{p|n} \left(1 - \frac{1}{p}\right)$$



Just as with the divisor function, it becomes much easier to compute the totient of n when we know its prime factorization. The product is over all prime factors of n . We consider an example.

Example 31 Consider $n = 120 = 2^3 3^1 5^1$. Then $\varphi(120) = 120 \left(1 - \frac{1}{2}\right) \left(1 - \frac{1}{3}\right) \left(1 - \frac{1}{5}\right) = 32$.

We now give a brief derivation of the formula just used. If the prime factorization of n is $p_1^{a_1} p_2^{a_2} p_3^{a_3} \dots p_k^{a_k}$, then

$$\varphi(n) = \prod_{j=1}^k \left(p_j^{a_j} - p_j^{a_j-1}\right) \tag{2}$$

$$= \left(p_1^{a_1} - p_1^{a_1-1}\right) \left(p_2^{a_2} - p_2^{a_2-1}\right) \dots \left(p_k^{a_k} - p_k^{a_k-1}\right) \tag{3}$$

$$= p_1^{a_1} p_2^{a_2} \dots p_k^{a_k} \left(1 - \frac{1}{p_1}\right) \left(1 - \frac{1}{p_2}\right) \dots \left(1 - \frac{1}{p_k}\right) \tag{4}$$

$$= n \prod_{p|n} \left(1 - \frac{1}{p}\right) \tag{5}$$

In eq 5, the product ranges over the distinct prime factors of n . This equation is often the more convenient one to use for hand calculation and, indeed, very convenient for spreadsheet implementation, as outlined in section 6.

Figure 4.6 shows the graph of totient for $1 \leq n \leq 50$. The straight line is $y = x - 1$ since the totient is maximum at the primes, p , where it takes values $p - 1$.

4.7 The Möbius function $\mu(n)$

Definition 32 If n is a positive integer, we define the Möbius function $\mu(n)$ as follows.

$$\mu(n) = \begin{cases} 1 & \text{if } n \text{ has an even number of distinct prime factors} \\ -1 & \text{if } n \text{ has an odd number of distinct prime factors} \\ 0 & \text{if } n \text{ is divisible by the square of a prime} \end{cases} \quad (6)$$

Remark 33 $\mu(1) = 1$ since the number 1 has no prime factors and 0 is even.

Example 34 $\mu(50) = 0$, since 50 is a multiple of 5^2 and 5 is prime. $\mu(15) = 1$ since $15 = 3 \times 5$ (2 factors). $\mu(30) = -1$, since $30 = 2 \times 3 \times 5$ (3 factors).

Remark 35 $\mu(n)$ is rather trivial to compute if we have access to the unique prime factorization of n . Suppose $n > 1$ is given by

$$n = p_1^{a_1} p_2^{a_2} p_3^{a_3} \dots p_k^{a_k} = \prod_{j=1}^k p_j^{a_j}$$

Then, if every $a_j = 1$, $\mu(n) = (-1)^k$, otherwise $\mu(n) = 0$. Another way of defining this function is to say that if $n > 1$ has a repeated prime factor, then the function is 0; otherwise its value is 1 or -1 depending on whether it has an even or odd number of distinct prime factors respectively.

Theorem 36 Let m, n be coprime integers, i.e., $\gcd(m, n) = 1$. Then $\mu(mn) = \mu(m)\mu(n)$.

Definition 37 *Multiplicative function.* A function $f(n)$ defined on the positive integers is multiplicative if $f(nm) = f(n)f(m)$ whenever n and m are relatively prime. It follows that $f(1) = 0$ or $f(1) = 1$. If the former is true, then the function is always zero, so we normally require that $f(1) = 1$.

The value of a multiplicative function $f(n)$ can be determined from its values at distinct primes. Thus, if

$$n = p_1^{a_1} p_2^{a_2} p_3^{a_3} \dots p_k^{a_k} = \prod_{j=1}^k p_j^{a_j}$$

then

$$f(n) = f(p_1)^{a_1} f(p_2)^{a_2} f(p_3)^{a_3} f(p_k)^{a_k} = \prod_{j=1}^k f(p_j)^{a_j}$$

In the following sections we describe how to construct an Excel model to compute the prime factorization of integers up to 30,000. The model also presents recursive techniques to accumulate the basic three functions of number theory, $\tau(n)$, $\varphi(n)$, $\mu(n)$ as defined in sections 4.4, 4.6, 4.7.

Remark 38 All three of the basic number-theoretic functions (divisor, totient and Möbius) are multiplicative. This property is essential for the application of totient to the RSA cryptosystem.

5 Spreadsheet techniques for illustrating the basic functions

By invoking the power of VBA, which sits behind Excel, we may write procedural code for $\tau(n)$, $\varphi(n)$, $\mu(n)$ computed directly from the definitions. In this manner, using naive but correct algorithms, each of these three functions may be implemented as a VBA function in only a few lines of code. If the teacher wishes to take this approach, then the students must have some understanding of elementary VBA and in particular are comfortable with user-written functions and simple forloops. In this case, our approach is to first present the code. Following this, an implementation with recursive spreadsheet formulas is considered. The students are then encouraged to compare implementation methods and perhaps even to come up with improved approaches. By presenting the material in this manner, we claim that pedagogy is served better than by the traditional approach of definition, theorem, proof, definition, theorem, proof, corollary, etc. Both the VBA and recursive approaches are described in the following two sections.

5.1 Recursive implementation in Excel

We now present an alternative implementation, using only the spreadsheet functions IF, MOD and SUM. It is instructive to consider how one might implement the divisor, totient and Möbius functions in Excel, and without any VBA code. The model also generates the unique factorization for any integer up to the maximum limit. The first step is to decide upon the maximum prime for the model; we choose $p_{\max} = 9973$. The user may use a slider (scrollbar) to choose any integer from 2 to p_{\max} and then model will generate the three functions and also the prime factorization of the given number. When the unique prime decomposition of n is available in Excel, then it becomes a rather straightforward matter to compute these functions in the spreadsheet environment. A recursive approach is used, based on a list of primes from 2 up to $p_{\max} = 9973$. The model may be used for rather large numbers as long as they don't have a prime factor larger than 9973, or any prime factor repeated more than 20 times. The model automatically checks for the first possibility, but not the second.

6 Advanced Excel model to generate prime factorization

*Note: if you wish to avoid the first five steps here (just below), you can paste the primes you need at **A3:A1231** in your worksheet from <http://primes.utm.edu/lists/small/1000.txt> [12], and skip straight to step 6.*

1. In a fresh Excel workbook, and on Sheet1, switch to VBA, by ALT-F11 and insert a new module. You need to have the Developer tab showing on the ribbon. If it isn't, click on the Office button and then Excel Options at the bottom of the dialog that appears. Click on Popular options and then "show developer tab in the ribbon". Close the dialog. Then type or paste in the following code:

```

Option Explicit
Public Function tau(n As Long) As Long
Dim k As Long
    tau = 0
    For k = 1 To n
        If n Mod k = 0 Then tau = tau + 1
    Next k
End Function

```

2. Save your file as primes.xlsm. Note the xlsm extension.
3. The code entered in step 1 allows us to easily generate all primes up to 9973. Return to Excel with ALT-F11 and put the integers 2 through 10000 in A1 through A9999. A convenient way to do this is to first put 2 in A1, click elsewhere, then click back in A1 again. Then go Fill Series... on the Editing panel of the home tab of the ribbon. Make sure you specify "Series in columns", Step value 1 and Stop value 10000. Click OK to dismiss the dialog box. You should have the integers 2 through 10000 in A2:A9999. You can check this by going CTRL down arrow while still in column A. The cursor should jump to A9999 where you will see the end of the series. Then insert the formula =tau(A1) in B1 and double-click B1's fill-handle. You should find that column B now has the number of factors for each of the integers from column A.
4. To isolate the primes from column 1 we need to identify those numbers with just two factors. This is easy, by sorting the two columns using column B as the key. First, select your data by clicking in A1, release the mouse button, then hold down CTRL-SHIFT while pressing RIGHT-ARROW then DOWN-ARROW keys. Your selection should now be the range A1:B9999. Click on the Data tab of the ribbon and then click on Sort. Choose to sort by Column B, then click OK.
5. All of your primes, which each have 2 factors, will now be at the top of your list in column B. Simply select the primes from A1 to A1229 and paste special these numbers as values in A3:A1231 of Sheet2. Now delete Sheet1. We now have all the primes we need in column A of a fresh worksheet.
6. The number for which we compute the various number-theoretic functions and which we shall factor is denoted n . It occupies cell A1. Click in A1 and then click in the name box and type n followed by the enter key.
7. Now fill the range B1:U1 with the integers 1 through 20. With the range B1:U1 selected, type the name k in the name box, followed by the enter key. These numbers, collectively known as k , represent the possible multiplicities of prime factors, collectively p . For example, if $x = 162$, then $x = 2^1 \times 3^4$, with one copy of

the prime $p_1 = 2$ multiplied by four copies of the prime $p_2 = 3$. Multiplicities here are $k_1 = 1$ and $k_2 = 4$. We may write the unique factorization as $x = p_1^{k_1} \times p_2^{k_2}$.

8. Make the columns B through U much narrower (perhaps 20 or so pixels wide), and centre the cells B:U (the entire columns B through U).
9. Select all the primes and name them collectively p. Probably the simplest way to do this is to click in A3, then CTRL-SHIFT DOWN-ARROW, then type the name p in the name box, followed by the enter key.
10. We now enter our prototype formula which is the heart of the model. Its sole purpose in life is to tell us whether p^k is a divisor (factor) of n . Click in B3 and enter the formula =IF(MOD(n,p^k)=0,1,""). This formula simply generates a one if p^k is a divisor (factor) of n and nothing (empty string) otherwise.
11. Drag the fill-handle of B3 across to U3. This will replicate the formula with no need for messy, partially-absolute references as we have used *names*. With the range B3:U3 still selected, double-click the fill-handle of U3. Our prototype formula should now have filled the range B3:U1231. You should now be looking at a vast array of ones!
12. To clearly delineate this array place a single-line border around B3:U1231. Do the same for V3:V1231. Select the range B2:V2 and format it as "Merge & Center". Place the heading "factors & their multiplicities" in this range.
13. Click in A1 and type the value 162. The spreadsheet landscape will now change to reveal the ingredients that will leads us to the unique prime factorization of 162. Wherever a one appears in the range B3:U1231, we have a power of some prime as a factor of n . The prime will be in column A and the power will be the value of k corresponding to the last column containing a one. For $n = 162$, you should see ones in just B3 and B4:E4, corresponding to the factorization $162 = 2^1 \times 3^4$.
14. Now put the heading **count** in V1.
15. Click in V3 and then click on the *autosum* button \sum (top right of the Home tab on the ribbon). Excel will propose to add the contents of the range A3:U3; you need to correct this to B3:U3. Then, double-click the fill-handle of V3 to replicate this formula through V4:V1231. Column V has the multiplicities of each of the primes in the unique factorization of x . Most are zero. We can tidy up our model by using conditional formatting to hide the zeroes. While the range V3:V1231 is still selected, click the conditional formatting button on the Home tab of the ribbon, and then click on New Rule ... In the box, "Select a Rule Type", click on "Format only cells that contain". Set up cell value equal to 0, then click on the Format button. Go for the Font tab and the select colour white. Then click OK twice to close the dialogs. You should now find that only non-zero entries appear in column V.

16. It's a good idea to check that the factorization is correct. Place the heading **build n** in **W1** and the value 1 in **W2**. Then in **W3**, type the formula `=W2*p^V3` and double-click **W3**'s fill handle to replicate this formula down the column.
17. We now use our model to compute the divisor function $\tau(n)$. In **X1**, place the heading **build tau(n)**. We defined this function in section 4.4 and expressed it in terms of the multiplicities of primes occurring the prime factorization of n in eq 1. From this equation, it is seen that we may build up the tau function by starting our running product at 1 (empty product) and then, each time we encounter another prime factor, p_j , of multiplicity a_j , multiply our running product by $(a_j + 1)$. To implement this recurrence in Excel, first, place 1 in **X2**. Then, in **X3**, type the formula `=X2*(V3+1)`. Replicate this formula down column **X**.
18. A similar approach is used to develop the Euler totient function $\varphi(n)$. In **Y1**, place the heading **build phi(n)**. We defined this function in section 4.6 and expressed it in terms of a simple iterated product whose factors correspond to just those primes occurring in the unique prime factorization of n (eq 5). From this equation, it is seen that we may build up the phi function by starting our running product at n and then, each time we encounter another prime factor, p multiply our running product by $(p - 1) / p$. To implement this recurrence in Excel, first, place the formula `=n` in **Y2**. Then, in **Y3**, type the formula `=IF(V3=0,Y2,Y2*(p-1)/p)`. Replicate this formula down column **Y**.
19. Our final number-theoretic function is that of Möbius, named $\mu(n)$. Again, we use a recursive approach to compute this function. Since $\mu(1) = 1$, we begin by placing the heading **build Möbius** in cell **Z1** and the value 1 in **Z2**. We defined $\mu(n)$ in eq 6, and we see from this definition that, going down the column, no change is necessary unless a prime factor is found. In this case, if the new prime factor occurs with multiplicity just 1, we set the current column value to the negation of the previous column value, else 0 (multiplicity more than 1). The formula which implements this logic, to be placed in cell **Z3**, and replicated down the column, is `=IF(V3=0,Z2,IF(V3=1,-Z2,0))`.
20. We now extend the model to generate the unique prime factorization of n . In **AA1** put the heading **(factor, multiplicity)**. We wish to generate, in column **AA**, ordered pairs (p_j, a_j) such that the prime factorization of n has exactly a_j prime factors equal to p_j . The prototype formula to accomplish this for us is `=IF(V3=0, "", "(" & p & ", " & V3 & ")")`, to be placed in cell **AA3**. Replicate this formula down the same column, by double-clicking **AA3**'s fill-handle.
21. Our purpose now is to combine the factors found in the previous step. We adopt a common format: for example, the factorization $162 = 2^1 \times 3^4$ is represented as the set of ordered pairs $\{(2, 1), (3, 4)\}$. This is the format adopted by, for example, Wolfram's *Mathematica*. It is normally assumed that the prime factors in such a representation appear in ascending order, but that is not absolutely essential. To

take initial steps to generate such a format in Excel, first place the heading **build prime decomp** in AB1. For reasons which will soon emerge, then place the formula ="" (empty string) in AB2. This initiates our recursion. Then, in AB3, type the formula =IF(V3=0,AB2,AB2 & AA3 & ", "). Fill-down.

22. Provided our input value of n in A1 has no prime factors exceeding 9973, cell AB1231 will contain the prime decomposition of n with an extraneous comma appended to it. So, in AC1, put the heading **Prime decomposition**, then in AC2, type the formula ="{" & LEFT(AB1231,LEN(AB1231)-2) & "}". This simply gets rid of the trailing comma and blank from AB1231, while enclosing the final string in a pair of braces to denote a set of pairs. You may have to widen column AC somewhat.
23. The functions $\tau(n)$, $\varphi(n)$, $\mu(n)$ may now be shown in columns AD through AF. In AD1 put the heading **tau(n)**, in AE1 the heading **phi(n)**, and in AF1 the heading **Möbius(n)**. Provided n has no prime factor greater than p_{\max} , the values of these three functions will appear in X1231 through Z1231. It is convenient to copy these values to AD2 through AF2. The simplest way to do this is to type the formula =X1231 in cell AD2 and then drag AD2's fill-handle across to AF2.
24. Although not absolutely essential, a nice touch for our model is to suppress the display of essentially redundant information. We follow similar steps as in step 15 to suppress repeated values in columns X through Z and AB. We describe the steps for column X. First, click in X3. Open the conditional formatting dialog again with "New Rule". We wish to display values in column X only if it differs from the previous value in that column. So we select "Use a formula to determine which cells to format". In the "Format values where this formula is true:" edit box, click in cell X3 and hit F4 twice to get "\$X3". This is a partially-absolute reference to X3 which will allow relative referencing with respect to rows. Then type "=" and click in X2. Again, F4 twice to get "\$X2". Click on the Format button, and again go for the white font. Double-clicking X3's fill-handle will now propagate X3's formula and formatting rule down the column. You could also use the formatting paintbrush to apply X3's format to the rest of the column, but double-click of fill-handle is easier. Assuming you still have 162 in A1, you should now see the contents of X5 and downwards vanish! In fact, they are still present, and you can check this by either double-clicking in X5 or clicking in X5 and then checking out the formula bar. We have simply asked for a white font with a white background.
25. The simplest way to format columns Y, Z, AB in the same way is to first click in X3, then click on the format paintbrush. This picks up the formatting rules from X3. Then click in Y3 to apply that format. Double-clicking Y3's fill-handle will then cause the formatting rule to be propagated down the column, as we did for the X column. Do the same for columns W, Z and AB.
26. We include a little check in our model just in case the user has entered a value of n in A1 which has prime factor exceeding p_{\max} . In AG1, put the heading "Correct?"

3441	Prime decomposition	tau(n)	phi(n)	Möbius(n)	Correct?
	{{3, 1}, (31, 1), (37, 1)}	8	2160	-1	TRUE
2					
3					
5					
7					
11					
13					

Figure 2: Advanced Excel model for factorization.

and in AG2 the formula =n=W1231. Finally, use conditional formatting in AG2 to alert the user if the model's results are incorrect (this means that n contains a prime factor greater than p_{\max}).

27. A slider is now added to make it easier for the user to alter the input value n . Click on the developer tab of the ribbon. Then click on Insert, Form controls, Scrollbar; drag to create a scrollbar of desired size somewhere near AC5. Right-click on this scrollbar and select Format control. Then set minimum value to 1, maximum value to 30,000, and cell link to A1. Click OK to close the dialog and your model is finished.
28. We have found it useful to hide computational details of the model from elementary classes. If this is desired, select columns B through Z and right-click, then Hide. A snapshot is shown in Figure 2.

7 Acknowledgement

The authors wish to thank the anonymous reviewers for many very helpful comments and suggestions. The second author would like to acknowledge West Virginia University for the award D3-10-040 that supported this work.

8 Exercises

1. Students can use the slider to find the prime factorization of numbers. By using the slider on the prime factorization spreadsheet, find the prime factorizations of
 - (a) 2241
 - (b) 4419
 - (c) 8288
 - (d) 9996

2. Students can put positive integers directly into the prime factorization spreadsheet to find the prime factorization. Find the prime factorizations of 90447, 146475, 209304088 by putting these values into the Excel model for prime factorization.
3. Explain how to identify primes using the model of section 4.3.2.
4. Extend the model of section 4.3.2 to count the number of factors of n . Be careful to cater for the case where n is a perfect square.
5. Determine the values of the three number-theoretic functions at the four numbers in Exercise 1 by using the spreadsheet and formulas.
6. Determine the values of the three number-theoretic functions at the three numbers in Exercise 2 by using the spreadsheet and formulas.
7. Investigate the values of $\tau(p^2)$ where p is prime using the spreadsheet model of section 6. Can you show, using the formula for τ , that your observation is true for all squares of primes?
8. Investigate the value of $\tau(p^3)$ where p is prime using the spreadsheet model. Can you establish that your observation is true for all cubes of primes? Why doesn't the result work for non-prime numbers?
9. Show that if n is "square-free", i.e., it has no repeated prime factor, then $\tau(n) = 2^k$, for some number k . Hint: look at the form of the unique prime factorization of n . How is the number k related to this factorization? Outline a method for systematically enumerating, or listing all of the factors of n in this case.
10. The spreadsheet model gives the prime factorization of all numbers n expressed in the form $p_1^{a_1} p_2^{a_2} p_3^{a_3} \dots p_k^{a_k}$ where each prime p_i does not exceed the biggest four-digit prime 9973. List four positive integers that contain 9973 as a prime factor. How many positive integers exist containing 9973 as a prime factor? Why?
11. Use the advanced model to compute $\text{gcd}(12960000, 250047000)$.
12. Use the advanced model with $n = 2^{21}$. Why does the model fail?
13. We defined two numbers m and n to be relatively prime if $\text{gcd}(m, n) = 1$. We may write their prime factorizations as $m = p_1^{a_1} p_2^{a_2} p_3^{a_3} \dots p_k^{a_k}$ and $n = q_1^{b_1} q_2^{b_2} q_3^{b_3} \dots q_l^{b_l}$ where $1 \leq k, 1 \leq l$ are $p_1, p_2, p_3, \dots, p_k$; $q_1, q_2, q_3, \dots, q_l$ are distinct primes. Investigate the following claim: "*For each positive integer j , $2j + 1$ and $4j^2 + 1$ are relatively prime.*"
14. We call the numbers $2^n - 1$ for n a positive integer, *Mersenne numbers*. Construct a spreadsheet model that calculates the first 20 Mersenne numbers. From your model, make a conjecture about when Mersenne numbers are prime. Extend your model to determine if your conjecture holds for the first 50 Mersenne numbers. What about the first 100 Mersenne numbers?

15. Use the Excel factorization model to write down the values for $\varphi(2)$, $\varphi(4)$, $\varphi(8)$, $\varphi(16)$, $\varphi(32)$, $\varphi(64)$, $\varphi(128)$ and $\varphi(256)$. What is the value of $\varphi(2^n)$? What is the value of $\varphi(4^n)$?
16. Use the Excel factorization model of section 6 to compute $\varphi(2) \times \varphi(9)$ and $\varphi(3) \times \varphi(6)$. Compare these with the value of $\varphi(18)$. Is it true that $\varphi(2) \times \varphi(9) = \varphi(18)$? What about $\varphi(3) \times \varphi(6) = \varphi(18)$? Investigate for what values of positive integers a and b , $\varphi(a) \times \varphi(b) = \varphi(a \times b)$. Can you formulate a conjecture here? Can you prove it?
17. The "big omega function" $\Omega(n)$ sums the multiplicities of the prime factors of n . For example, since $360 = 2^3 3^2 5^1$, $\Omega(360) = 3 + 2 + 1 = 6$. Extend the model of section 6 to implement this function.
18. The "little omega function" $\omega(n)$ is the total number of different prime factors of n . For example, since $360 = 2^3 3^2 5^1$, $\omega(360) = 3$. Also, $\omega(p) = 1$ for all primes p . Extend the model of section 6 FTA to implement this function.

9 Solutions to exercises

1. (a) $2241 = 3^3 83^1$
 (b) $4419 = 3^2 491^1$
 (c) $8288 = 2^5 7^1 37^1$
 (d) $9996 = 2^2 3^1 7^2 17^1$
2. $90447 = 3^1 7^1 59^1 73^1$, $146475 = 3^3 5^2 7^1 31^1$, $209304088 = 2^3 7^3 83^1 919^1$
3. The only CF highlight is in the first row where $x = 1$.
4. This is tricky. We need to count the number of zeros in column E but only up to $x^2 \leq n$. We then double this figure, but need to subtract one if n is a perfect square since \sqrt{n} will have been counted twice. For the prototype formula in G2, there is nothing to count unless E2 is zero and $x^2 \leq n$. In such a case we then want 2 if $x^2 < n$ and 1 if $x^2 = n$. This leads to the prototype formula `=IF(AND(E2=0, x*x<=n), IF(x*x=n, 1, 2), 0)` in G2 which is then filled down to G32. A binary decision tree may be useful to graphically illustrate the logic for this problem. The final step is to place the heading # factors in H2 and the formula `=SUM(G:G)` in H2.

n	$\tau(n)$	$\varphi(n)$	$\mu(n)$
2241	8	1476	0
4419	6	2940	0
8288	24	3456	0
9996	36	2688	0

5.

n	$\tau(n)$	$\varphi(n)$	$\mu(n)$
90447	16	50112	1
146475	48	64800	0
209304088	64	88524576	0

6.

n	$\tau(n)$	$\varphi(n)$	$\mu(n)$
90447	16	50112	1
146475	48	64800	0
209304088	64	88524576	0
7. $\tau(p^2) = 3$ for any prime p .
8. $\tau(p^3) = 4$ for any prime p .
9. If n is "square-free", i.e., it has no repeated prime factor, then we may write

$$n = \prod_{k=1}^m p_k = \prod_{k=1}^m p_k^1$$

so that

$$\tau(n) = \prod_{k=1}^m (1 + 1) = 2^m$$

In this product, m is the number of distinct prime factors of n . These may be enumerated by using a truth table technique in Excel.

10. Four integers with factor 9973 are

$$\begin{aligned} 9973 \times 1 &= 9973 \\ 9973 \times 2 &= 19946 \\ 9973 \times 3 &= 29919 \\ 9973 \times 4 &= 39892 \end{aligned}$$

Infinitely many integers have 9973 as a factor. We simply extend the list above to generate, in principle, all multiples of 9973. By FTA, these are the only integers having 9973 as a factor.

- 11.

$$\begin{aligned} 12960000 &= 2^8 3^4 5^4 \\ 250047000 &= 2^3 3^6 5^3 7^3 \\ \gcd(12960000, 250047000) &= 2^3 3^4 5^3 \\ &= 81000 \end{aligned}$$

12. The largest prime multiplicity (frequency) allowed by the model is 20.
13. An Excel model tabulating these three quantities yields $\gcd(j, 2j + 1, 4j^2 + 1) = 1$ for $1 \leq j \leq 1000$. Thus, the claim appears to be true.
14. See http://en.wikipedia.org/wiki/Mersene_prime.

15. $\varphi(2) = 1$ $\varphi(4) = 2$
 $\varphi(8) = 4$ $\varphi(16) = 8$
 $\varphi(32) = 16$ $\varphi(64) = 32$
 $\varphi(128) = 64$ $\varphi(256) = 128$
 $\varphi(2^n) = 2^{n-1}$ $\varphi(4^n) = 2^{2n-1}$
16. $\varphi(a) \times \varphi(b) = \varphi(a \times b)$ if a and b are coprime. $\varphi(2) \times \varphi(9) = 1 \times 6 = 6$; $\varphi(3) \times \varphi(6) = 2 \times 2 = 4$; $\varphi(18) = 6$
17. The formula =SUM(count) added to the advanced model will compute the "big omega function".
18. The formula =COUNTIF(count, ">0") added to the advanced model will compute the "little omega function".

10 Conclusion

We have presented a powerful model in Excel for generating the unique prime factorization of integers up to 30,000 using the slider and larger numbers if put in manually. The model may be used to illustrate the very basics of number theory. It allows for considerable experimentation and investigation by students. Conjectures may be formed and tested very quickly in the flexible spreadsheet environment.

References

- [1] Abramovich, S. and Levin, I. (1993). Microcomputer-based discovery and testing of combinatorial identities. *Journal of Computers in Mathematics and Science Teaching*, 13(2), p. 223-246.
- [2] Abramovich, S. (1999). Revisiting an ancient problem through contemporary discourse. *School Science and Mathematics*, 99, p. 148 - 155.
- [3] Abramovich, S., and Sugden, S.J. (2004). Spreadsheet conditional formatting: An untapped resource for mathematics education, *Spreadsheets in Education (eJSiE)*: Vol. 1: Iss. 2, Article 3. Available at: <http://epublications.bond.edu.au/ejsie/vol1/iss2/3>. Accessed 2009-05-18.
- [4] Baker, J.E. (2007). Excel and the Goldbach Comet, *Spreadsheets in Education (eJSiE)*: Vol. 2: Iss. 2, Article 2. Available at: <http://epublications.bond.edu.au/ejsie/vol2/iss2/2>
- [5] Agargun, A.G., and Ozkan, E.M. (2001). A historical survey of the Fundamental Theorem of Arithmetic. *Historia Mathematica*, 28, p. 207-214.
- [6] Battista, M.T., Van, C., and Borrow, A. (1998). "Using spreadsheets to promote algebraic thinking" *Teaching Children Mathematics* 4: 470-478.

- [7] Beare, R. (1993). How spreadsheets can aid a variety of mathematical learning activities from primary to tertiary level. In B. Jaworski (Ed.), *Technology in Mathematics Teaching: A bridge between teaching and learning*. Birmingham.
- [8] Cairns, G. (2003). Is there a greater role for prime numbers in our schools? *Australian Senior Mathematics Journal*, v19 n1 p24–37 2005.
- [9] Devlin, K. (2009) What is algebra anyway? *The Celebration of Teaching & Learning*. <http://thirteencelebration.org/blog/edblog/what-is-algebra-anyway/1081/>. Accessed 2009-05-12.
- [10] Calder, N., Brown, A., Hanley, U., and Darby, S. (2006) Forming conjectures within a spreadsheet environment, *Mathematics Education Research Journal*, Vol. 18, No. 3, 100–116.
- [11] Calder, N. (2004). Shaping understanding: How does investigating in a spreadsheet environment affect the conversations of initial training students doing mathematics. In I. Putt, R. Faragher & M. McLean (Eds.), *Mathematics education for the third millenium: Towards 2010 (Proceedings of the 27th annual conference of the Mathematics Education Research Group of Australasia, Townsville, pp.143-150)*. Sydney: MERGA.
- [12] Caldwell, C. *The Primes Pages*. University of Tennessee at Martin. Accessed 2010-09-20. URL: <http://primes.utm.edu/>.
- [13] Funnell, L., Marsh, T., and Thomas, M. (1995). Strategies for integrating computers into mathematics lessons: emphasising spreadsheets - SAME papers. Hamilton: University of Waikato, Centre for Science, Mathematics, and Technology Education Research, (pp. 223 - 238).
- [14] Goldman, J. (2004). *The Queen of Mathematics: A historically motivated guide to number theory*, A K Peters, Ltd.
- [15] Liljedahl, P., Sinclair, N., & Zazkis, R. (2006). Number concepts with number worlds: Thickening understandings. *International Journal of Mathematical Education in Science and Technology*, **37**(3), 253–275.
- [16] Lim, K. (2004). A Survey of First-Year University Students' Ability to use Spreadsheets, *Spreadsheets in Education (eJSiE)*: Vol. 1: Iss. 2, Article 1.
- [17] Hardy, G.H. and Wright, E.M. (1995). *An introduction to the theory of numbers, fifth edition*. Clarendon Press, Oxford.
- [18] G. H. Hardy (1940). *A mathematician's apology*, Cambridge University Press.
- [19] Haspekian, M. (2003). Between arithmetic and algebra: a space for the spreadsheet? Contribution to an instrumental approach. *Proceedings of the third Conference of the European Society for Research in Mathematics Education*. Pisa: Universita di Pisa.

- [20] Ploger, D., Klinger, L., and Rooney, M. (1997). Spreadsheets, patterns, and algebraic thinking. *Teaching Children Mathematics* **3**: 330–334.
- [21] Sugden SJ (2007). Spreadsheet Investigations in Modular Arithmetic. *Mathematics into the 21st Century Project. Proceedings*, 625–630. Charlotte, North Carolina.
- [22] Sutherland, R., and Rojano, T. (1993). A spreadsheet approach to solving algebra problems. *Journal of Mathematical Behaviour* **12**(4): 351–383.
- [23] Van Lint, J.H. (1994). What is discrete mathematics and how should we teach it? In *Selected Lectures from the 7th International Congress in Mathematical Education*, edited by D.F. Robitaille, D.H. Wheeler, and C. Kieran (Les Presses de l'Universite Laval), 263–270.
- [24] Wilson, K.E., Ainley, J., and Bills, E. (2005). Designing spreadsheet-based tasks for purposeful algebra, *International Journal of Computers for Mathematical Learning*, **10**, 3, 191–215.
- [25] Wilson, K.E., Ainley, J., and Bills, E. (2006). Spreadsheets, pedagogic strategies and the evolution of meaning for variable, In Chick, H. L. & Vincent, J. L. (Eds.). *Proceedings of the 29th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 4, pp. 321-328. Melbourne: PME.
- [26] http://en.wikipedia.org/wiki/Fundamental_theorem_of_arithmetic#Proof. Accessed 2010-07-02.