**Spreadsheets in Education (eJSiE)**

Volume 3 | Issue 3 Article 5

1-23-2010

# Spreadsheets and the development of skills in the STEM disciplines

Sergei Abramovich
*State University of New York at Potsdam*, abramovs@potsdam.edu

Galina V. Nikitina
*St. Petersburg Branch of the Academy of Information Technologies in Education, RUSSIA*, moikariver@yandex.ru

Vladimir N. Romanenko
*St. Petersburg State University of Technology and Design, RUSSIA*, ladogalake@gmail.com

# Spreadsheets and the development of skills in the STEM disciplines

**Abstract**

The paper describes a number of practical uses of a spreadsheet that vary across the novice-expert continuum in terms of the development of skills required for career paths in the STEM (science, technology, engineering, mathematics) disciplines. To this end, three types of skills are introduced: basic, professional and advanced. Using a spreadsheet as a background and Vygotskian perspective on learning and development as a conceptual framework, the paper demonstrates how purposeful applications of the skills transform one type into another, encourage concept learning, and broaden technology integration into problem solving. The relationship between a spreadsheet and other software tools is discussed.

**Keywords**

Spreadsheet, Skills, Inequalities, Proof, Modeling

# Spreadsheets and the development of skills in the STEM disciplines

*Sergei Abramovich*
*State University of New York at Potsdam, USA*
*abramovs@potsdam.edu*

*Galina V. Nikitina*
*Academy of Information Technologies in Education*
*St. Petersburg, RUSSIA*
*moikariver@yandex.ru*

*Vladimir N. Romanenko*
*St. Petersburg State University of Technology and Design, RUSSIA*
*ladogalake@gmail.com*

March 25, 2010

**Abstract**

The paper describes a number of practical uses of a spreadsheet that vary across the novice-expert continuum in terms of the development of skills required for career paths in the STEM (science, technology, engineering, mathematics) disciplines. To this end, three types of skills are introduced: basic, professional and advanced. Using a spreadsheet as a background and Vygotskian perspective on learning and development as a conceptual framework, the paper demonstrates how purposeful applications of the skills transform one type into another, encourage concept learning, and broaden technology integration into problem solving. The relationship between a spreadsheet and other software tools is discussed.

Submitted January 2010, revised and accepted March 2010.

**Key words:** spreadsheet, skills, inequalities, proof, modeling.

## 1 Introduction

The main goal of learning is to acquire knowledge and to develop various skills. For example, learning to become a teacher requires considerable content knowledge and numerous grade appropriate pedagogical skills [24]. Learning to become a mathematician requires a broad knowledge of problem-solving skills that employ "only argument and computation" [5, p2]. Learning to become a scientist or engineer requires knowledge of how to interpret the world around us in mathematical terms [23] and possession of skills enabling one to move from novice practice to expert practice in solving problems [28]. Nowadays, future teachers, mathematicians, scientists, engineers, and other professionals need skills in acquiring knowledge through the use of digital technology that has dramatically changed learning environment at all levels of education and across all disciplines [20].

A classic example of modern technology is an electronic spreadsheet. It has been more that two decades since the tool has been used in the teaching of mathematics [6], engineering [28], and science [12]. Nowadays, the software is so widespread that even in many entry-level positions for high-school graduates, skill in creating and operating spreadsheets is a requirement [10]. This skill includes knowledge of many basic techniques such as storing, processing, and representing data within a spreadsheet. Beyond those basics, a spreadsheet, when used appropriately, can provide learners with much needed experience in mathematical modeling—a transferrable skill [19] that can be applied in a variety of advanced professional setting such as teaching, doing mathematics research, and solving engineering problems. Using various spreadsheet-based modeling techniques as a background, this article will illustrate and conceptualize the interplay between the acquisition of knowledge and the development of skills typically encountered in the context of formal schooling at the elementary, secondary, and tertiary levels [2].

## 2 Three levels of skills

A number of interesting patterns can be identified when knowledge and skills are conceptualized as a dual pair. Firstly, in order to acquire knowledge, one has to possess learning skills; put simply, one should be able to learn. These skills will be referred to below as *basic* skills. Secondly, any new knowledge has the potential to result in the development of skills that bear a professional flavor. Once basic skills reach a certain level of maturity, they can be used in applications. Skills used in applications will be referred to as *professional* skills. When used repeatedly, professional skills begin affecting individual abilities of the learner by changing and transforming these abilities. Such a chain of transformations in the development of basic skills is inherent to the whole process of education. Thirdly, regardless of their level, most of the basic skills are interdisciplinary in nature. The ability to extend the application of basic skills from one context to another can turn basic skills into what can be referred to as *advanced* skills. A purposeful application of advanced skills in different contexts can serve as a foundation for the development of new basic skills. By the same token, basic skills can become advanced skills but at a higher cognitive level. In that way, three kinds of skills will be considered in this article: basic skills, professional skills, and advanced skills.

It should be noted that what can be considered as advanced skills at one developmental level, can be viewed as basic skills at another level. For example, through learning to count a child develops basic counting skills. These skills can be applied with a purpose to a variety of contexts as professional skills (e.g., counting pennies when paying for a candy). Repeated use of basic counting skills in multiple contexts as professional skills

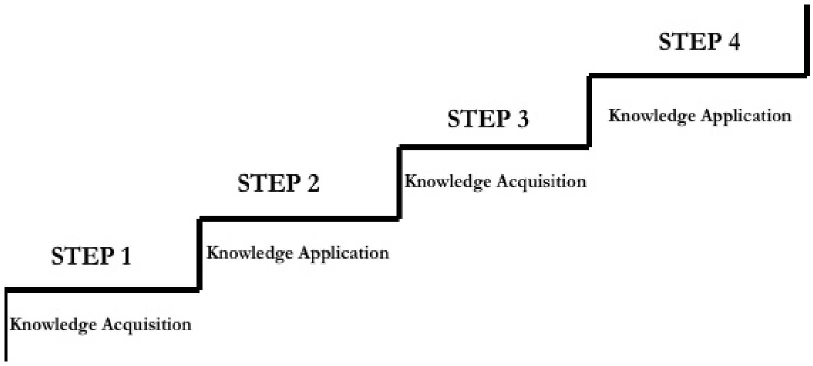



Figure 1: K$^2$A$^2$(knowledge acquisition–knowledge application) staircase.

elevates them to a certain level of maturity where they can be used as advanced skills (e.g., counting quarters and multiplying twenty five by the number of quarters when paying for an ice cream). By the same token, the repeated use of advanced skills in various contexts transforms them into basic skills at a higher developmental level.

Indeed, like a lower concept can be interpreted as a rudiment of a higher concept (e.g., the sequence of counting numbers is a special case of an arithmetic series), a basic skill can always be interpreted as a rudiment of the corresponding advanced skill. It is through the purposeful and repeated use of basic skills in different contexts that the child (or any individual for that matter) assimilates them. This assimilation of basic skills allows for their elevation to a new level of maturity from which one starts searching for a new purpose and context in order to apply the skills [17]. This is the case when by counting the number of intersections of two graphs as a way of deciding the number of real roots of the corresponding algebraic equation one uses basic counting skills and, in doing so, develops an advanced skill of geometrizing a mathematical concept. Thus, the level of maturity of skills is characterized by one's ability to transform them into a problem-solving tool [9]. As one later becomes a teacher, mathematician, scientist, or engineer, many basic skills developed at a certain level, transform into advanced skills and then back into basic skills but at a higher developmental level. It is through such changes in the description of skills that in a professional setting "the new higher concepts in turn [have the potential to] transform the meaning of the lower" [26, p202].

Metaphorically speaking, the cognitive development of an individual can be conceptualized as a movement along a staircase in such a way that if knowledge acquisition occurs at an odd step, then the application of knowledge occurs at an even step (Figure 1). In other words, basic skills are used at an odd step (to acquire knowledge) and professional and advanced skills are used at an even step (to apply knowledge) of such $K^2A^2$ (knowledge acquisition–knowledge application) staircase.

# 3 Illustration 1: developing a roster within a spreadsheet

Consider the case when one uses a spreadsheet to develop a roster with 30 names so that to each name a counting number has to be assigned. The task in question is to generate the first 30 counting numbers within a spreadsheet. Such quite a common use of the software can be conceptualized as a move along the $K^2A^2$staircase through which basic counting skills, mediated by the cells of a spreadsheet, are used as professional skills and then transformed into advanced skills. Being considered advanced skills for a novice practice they are transformed into basic skills of an expert practice. For the purpose of the analysis of the skills used, several approaches to the development of a roster by using a spreadsheet can be identified.

## 3.1 A non-computational approach with visual effects

One can enter the numbers one by one into a spreadsheet by typing 1, 2, 3, . . . , 30. This action can be seen as counting and labeling cells at the same time. What kind of basic mathematical skills are used here? Essentially, one uses knowledge of the order of numbers and their symbolic, base-ten representation. Whereas the former bit of knowledge is used in a conscious (though automatic) way, the second bit of knowledge is used unconsciously as the use of a spreadsheet does not require the conceptualization of the base-ten system in order to generate the numbers. Once numbers start entering the spreadsheet, basic counting skills become professional skills oriented towards the development of the roster.

The process of writing numbers one by one within the cells of a spreadsheet that mediate this process not only enhances counting skills but facilitates learning mathematical symbolism of the base ten system as well. Indeed, even by writing numbers "some general concept of the decimal system does develop" [27, p207]. However, according to Berg [7, p379],

*Vygotsky rightly points out that as long as one operates only with the decimal system without being aware of the other bases, one has not mastered the system; rather one is bound by it. When one learns other bases, one can consciously choose one's system and thus a new level of conceptual control is achieved.*

Therefore, a possible extension of this activity for a pure educational purpose is to learn counting in non-decimal bases within which the base number points at the number of digits in this base. Basic skills of counting in base-ten system can therefore be transformed into advanced skills enabling counting of objects in other base systems.

At that point, the following question arises: Is there a difference between a paper-and-pencil roster and a spreadsheet-based roster? One can see how being used in a purposeful context with reflection, basic counting skills can quickly reach an advanced level and, thereby, motivate the development of new knowledge about a spreadsheet. In turn, any new knowledge results in the emergence of new basic skills. The difference between on- and off-computer settings is that a spreadsheet can automatically change visual attributes of numbers (or, more generally, characters) including size, font, color, and alignment. The recognition of the advantage of using a spreadsheet transforms basic skills (via their use as professional skills) into advanced skills allowing one to control the visual representation of numbers, something that can be used later beyond the development of a roster as a new basic skill.

Indeed, through this task, a spreadsheet becomes an agency for the development of knowledge responsible for the emergence of advanced skills oriented towards the use of the software. Put another way, by using a spreadsheet to enhance visualization one gains experience in moving from novice practice to expert practice in operating the software. Through this process one develops certain rudiments of an engineering experience, important nowadays for preparing the STEM workforce of the future [18]. Also, one can start thinking if newly acquired knowledge and skills can be extended to other tasks where formatting plays an important role. Such a realization bears a potential for basic skills to be transformed into advanced skills to be used across different contexts.

The ease with which a large amount of data can at once be visually altered can motivate another inquiry into the capability of a spreadsheet: Can the required array of counting numbers be generated at once? That is, can one extend the use of the software from just enhancing visualization to enabling professionally designed computation? This inquiry into the capabilities of one of the most commonly used computer applications motivates a new transformational triad: from basic skills to professional skills to advanced skills.

### 3.2 A computational approach with mathematical applications

When entering counting numbers one by one into a spreadsheet, a user of the software does this automatically. Automatism, however, lacks conscious awareness – an important condition for creativity and insight. As Freudenthal [13] noted, "sources of insight can be clogged by automatisms" (p469). Though one acts consciously in entering numbers, his/her attention is not directed towards the conscious application of mathematical skills. Through a computational approach to the development of roster, one can become consciously aware of the following

**Property of counting numbers**. *Each counting number is one greater than the previous number and the first number is equal to one.*

That is, the use of a spreadsheet motivates the introduction of the recursive definition of counting numbers. Symbolically, this definition can be expressed in the form of the difference equation

$$\begin{aligned} &x_1 = 1 \\ &x_{n+1} = x_n + 1 \qquad\qquad n \geq 1 \end{aligned} \tag{1}$$

The recursive nature of counting numbers enables one to computerize definition 1 by using the so-called *ostensive* definition based on the pointing at the first term and defining the second term as the previous term plus one. In terms of cells and formulas they bear, by entering the unity in cell `A1`, defining the formula `=A1+1` in cell `A2`, and replicating the formula down the rows to cell `A30` one can generate the first 30 counting numbers electronically without the need to type them one by one.

Ironically, from a mathematical perspective, using formula 1, which was developed at a higher step of the $K^2A^2$ staircase in comparison with counting skills, reduces the need for using such lower level skills as knowing the order of numbers and their symbolic representation in the base-ten system. Yet, definition 1 gives new meaning to the counting skill by conceptualizing counting as a recursive process. For example, if one has lost count while counting (lined up) objects but remembers a number assigned to a certain object in the line, then one can continue counting from that object and not from scratch. In that way, learning to program a spreadsheet leads to the mastery of one of the basic concepts of discrete mathematics–the concept of recursion. As Vygotsky [27] would had have put it, conscious awareness of mathematics that underlie the counting skill "enters through the gate opened up by the scientific concept [of recursion]" (p191). This is just one example of how the use of a spreadsheet can play an important role in the development of mathematical knowledge that transforms basic counting skills into advanced skills allowing one to generate consecutive even numbers, odd numbers, and other arithmetic sequences as well as those of more complicated nature.

Moving up along the $K^2A^2$ staircase, a task of generating number sequences in non-decimal bases using a spreadsheet can be posed. At that level, an advanced skill of counting objects in non-decimal bases becomes a basic skill, which, in turn, can be transformed into an advanced skill of generating counting numbers, or more generally, arithmetic series in a symbolic form. It should be noted that the symbols 10, 20, 30, 40, and so on, acquire new meaning in non-decimal bases. For example, in base four the meaning of the first digit of a two-digit number 10 (it should not be read "ten" but rather one-zero-base-four) is one set of four ones (rather than one set of ten ones like in base ten).

Originally, an electronic spreadsheet was designed to use base ten numbers. In the case of a non-decimal base, the tool should be reprogrammed to record the results of addition through definition 1 differently. It should be noted that such reprogramming of a spreadsheet involves a rather lengthy formula using multiple functions `IF` which verify conditions formulated in terms of the function `INT`. The size of such a formula depends on the length of the string of digits representing a number. Because the use of a spreadsheet for doing arithmetic in non-decimal bases merits a separate paper, the authors chose not to discuss the formula here.

Figure 2 shows two spreadsheets that generated counting numbers in the bases ten (left) and four (right), respectively. Note in the latter base the only digits are $0, 1, 2, 3$. By changing the base number in a slider-controlled cell `B1`, one can generate consecutive counting numbers in multiple bases $B \leq 10$. For bases $B > 10$ one can use either letters or non-alphanumeric characters to represent digits beyond 0 through 9. This skill is especially important for those studying computer science.

## 3.3 The introduction of constraints

We now return to the base-ten system. The ease with which counting numbers were generated within a spreadsheet motivates another question: could this process be defined so that, in addition to generating counting numbers, the spreadsheet would automatically


Figure 2: Generating strings of the first ten counting numbers in bases ten and four.

stop displaying the numbers after the largest required number is displayed? Can the skill of counting up to a specific number be formalized using a mathematical definition? With this in mind, one can be introduced to the notion of constraint by adding to definition 1 an inequality which identifies the largest number (say, 30) to be displayed

$$\begin{aligned} x_{n+1} &= x_n + 1 \\ x_1 &= 1 \\ x_n &\leq 30 \end{aligned} \tag{2}$$

Once again, in order to be consciously aware of the meaning of the constraint, one has to computerize, or, put another way, to "teach" a spreadsheet to use the constraint. In the process of counting objects, what skill does one use in deciding when to stop labeling the objects with counting numbers? Perhaps unconsciously, before labeling a new object, one checks out the last assigned label, continues the process if this label is smaller than the constraining number and terminates counting otherwise. That is, at each step of counting, a current label should be compared to the number 30. Formally, such comparison consists in verifying if the current number is smaller than 30 – only in this case the process of counting continues. In other words, the comparison involves the use of an inequality. Such an inequality-based constraint represents a condition, which refines definition 2. One of the basic tools to introduce a value-restraining condition using spreadsheet syntax is the (conditional) function `IF`. This function has three parts: a condition, an action that must be taken if the condition is true, and an action that must be taken if the condition is false. Therefore, by defining `(A1)=1,(A2)=IF(A1<30,A1+1,“ ”)` and replicating cell `A2` down the rows at a sufficiently large distance from cell `A1` makes it possible to automatically control the length of the sequence of counting numbers.

Definition 2 can be generalized further to any arithmetic sequence of any given length as follows

$$x_{n+1} = x_n + d \tag{3}$$
$$x_1 = a$$
$$x_n \leq N \tag{4}$$

Using spreadsheet syntax, one has to do the following: `A1: =1`; cell `C1` is slider-controlled and given the name `length`, `(A2)=IF(A1 < length, A1+1," ")` and then replicate cell `A2` down column `A`. Figure 3 shows the string of the first ten counting numbers generated by the spreadsheet.

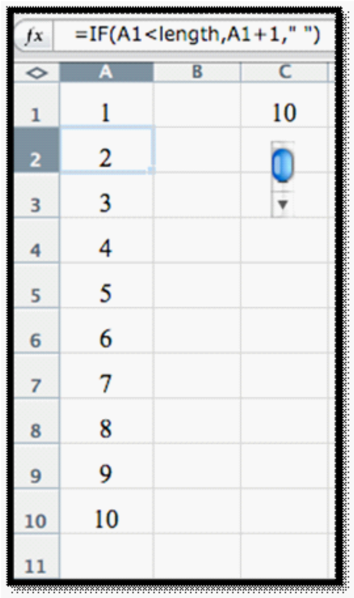



Figure 3: Generating the strings of the first counting numbers of length ten.

### 3.4 On the interplay between constraints and parameters

One can see that definition 1 is a special case of definition 3 with $a = d = 1$ and $N \in \mathbb{R}$. The latter inclusion means that $x_n$ can vary indefinitely without any constraint. On the other hand, when $d < 0$ and $N > a$, definition 3 does not include any constraint. This fact can be discovered by the repeated application of definition 3 for different values of parameters $a$, $d$, and $N$. So, the use of inequalities as constraints which specify the applicability of definitions can be seen as an advanced skill developed through encountering a limitation of a basic skill.

For example in order for constraint 4 to represent a meaningful (non-empty) condition, the following inequalities should hold true:

$$\text{if } d > 0 \text{ then } N \geq a \text{ (definition 3 generates an empty set of numbers if } N < a),$$
$$\text{if } d < 0 \text{ then } N \leq a \text{ (the constraint of definition 3 is meaningless if } N > a).$$

This inherent connection between a constraint (limitation) imposed on the behavior of a model and parameters of the model demonstrates the importance of constraints (limitations) as elements of any mathematical model. In the words of Aristotle, "the mathematical sciences particularly exhibit order, symmetry, and limitation; and these are the greatest forms of the beautiful " (cited in [11, p184]). The capability of a spreadsheet in the development of a skill in coordinating constraints and parameters towards the efficiency of a computational experiment should not be underestimated. By the same token, one's ability to understand a limitation of a tool, be it a mathematical model or a computational medium, should be recognized as another important professional skill. We will return to the discussion of the skill in using inequalities as means of providing the efficiency of spreadsheet-based calculations in the context of Illustrations 2 and 3.

### 3.5 Developing conditional formatting skill using a mathematical definition

Consider the task of generating a list of the first 30 counting numbers in which any two consecutive terms have different visual representations either in terms of font, or color, or size, and so on. What mathematical knowledge one needs to possess if this task is to be carried out in a paper and pencil setting? Besides skills already mentioned, one has, for example, to use different color pencils in writing two consecutive numbers. As was already discussed, in a spreadsheet environment one can alter visual representation of the whole string of numbers. But altering the appearance of numbers one by one is not much different than using colored pencils. How could one carry out this task by using a combination of mathematical knowledge and spreadsheet-specific skills? To answer this question, the idea of conditional formatting [3] can be introduced. Towards this end, counting numbers have to be defined by using their special properties. Obviously, between two consecutive counting numbers one number is even and another is odd. Knowing how to define these conditions in the syntax of a spreadsheet enables one to carry out the task in a very effective way.

More specifically, if we are to give red color to each even number within a string of counting numbers, the formula `=MOD(A1,2)=0` can be entered into the first conditional formatting dialog box. Consequently, the formula `=MOD(A1,2)>0` can be entered into the second conditional formatting box. This idea can be extended to assign red color to every third number by using the formulas `=MOD(A1,3)=0` and `=MOD(A1,3)>0` into the first and second conditional formatting boxes, respectively. The function `MOD(a,b)` returns the remainder when $a$ is divided by $b$.

Moving up along the $K^2A^2$staircase, one can use spreadsheet conditional formatting to assign red color to prime numbers only. This would require the use of more advanced skills in determining the minimum number of divisions to test for primality all numbers in a certain range. More specifically, by entering the formulas

$$\texttt{=OR(A1=2,A1=3,A1=5)} \quad \text{and} \qquad (5)$$
$$\texttt{=AND(MOD(A2,2)>0,MOD(A2,3)>0,MOD(A2,5)>0)}$$

as two conditions that assign red color to numbers that satisfy them, all prime numbers in the range $[1, 30]$ will appear in red. By learning to use conditional formatting facility of a spreadsheet, one develops conscious awareness of the following fact: it requires using prime numbers in the range $[2, \sqrt{N}]$ that in order to test if $N$ is a prime number or not. In that way, using the primes 2, 3, and 5 only, one can correctly decide if 47 is a prime number as 5 is the largest number smaller than $\sqrt{47}$. Once again, marking prime numbers with a pencil does not require conscious awareness of this property (rather, this can be done through memorization), whereas doing this through spreadsheet conditional formatting defined by conditions 5 does.

In the words of Vygotsky [27, p190],

*Lack of conscious awareness is not simply part of the conscious or unconscious. It does not designate a level of consciousness. It designates a different process in the activity of consciousness. I tie a knot. I do it consciously. I cannot, however, say precisely how I have done it. My action, which is conscious, turns out to be lacking in conscious awareness because my attention is directed toward the act of tying, not on how I carry this act. Consciousness always represents some piece of reality. The object of my consciousness in this example is the tying of the knot, that is, the knot and what I do with it. However, the actions that I carry out in tying the knot – what I am doing – is not the object of my consciousness. However, it can become the object of consciousness when there is conscious awareness. Conscious awareness is an act of consciousness whose object is the activity of consciousness itself.*

Here one can learn how to move from a novice practice to an expert practice through using the combination of mathematical knowledge and skills in operating a spreadsheet. This approach also plays and educative role: numbers defined through definitions 1–3 can be put in different groups depending on their properties. In that way, mathematical ideas and concepts can be introduced to learners in an applied context. In this context, basic mathematical skills, once being used as professional skills, become advanced spreadsheet programming skills.

## 4 Illustration 2: mathematical skills as tools in computing applications

The illustration of this section concerns the integration of spreadsheet modeling with the development of skills in constructing deductive proofs. For those majoring in mathematics, including prospective secondary mathematics teachers, one of the most important skills to be developed is that of articulating a formal argument. The development of this skill may occur through a practice in proving application-oriented mathematical statements using a spreadsheet as a milieu for conjecturing and proving [4]. In this illustration, the use of a spreadsheet is discussed in the context of learning to construct computationally efficient environments enabling numerical modeling of a two-dimensional problem formulated in the context of plane geometry.

In what follows, the skill of limiting a range for a variable involved through the use of inequalities that was previously discussed as an advanced modeling skill will be considered a basic skill. However, trying to extend the skill from one-dimensional to two-dimensional modeling situation, one can discover that this basic skill is insufficient as the ranges for two variables should be minimized and coordinated rather than just limited to a certain length. The new emphasis on the minimization of the (two-dimensional) domain of spreadsheet computations requires mathematical skill of extracting information about the mutual behavior of variables from the equation they are bounded to.

To this end, new reasoning skills have to be developed. The ability to recognize, conceptualize, and overcome a limitation of a basic skill that one possesses can be considered an advanced skill needed to move from novice practice to expert practice in spreadsheet modeling. Put another way, boundaries for the ranges of variables involved should be extracted from the information by using advanced mathematical reasoning skills.

Imagine a joint work of an engineer and mathematician on an applied problem: the former formulates the problem and the latter solves it using a variety of arguments and computational techniques. Through this process, the engineer can learn from the mathematician certain skills beyond the former's main area of expertise. By the same token, through solving engineering problems, the mathematician can appreciate how advanced mathematical skills develop in an applied context.

## 4.1 Thinking with a geometric figure as a skill

One of the earliest applications of mathematics was in the context of geometry. That is why the idea of thinking with a geometric figure and assigning geometric properties to numbers is known from the time of Pythagoras. A basic geometric exploration can easily be interpreted in terms of a practical activity. As an example of this kind of a mathematical learning environment, consider the following

**Brain Teaser**. A rectangle with whole number sides $x$ and $y$, $x > y$, has to be transformed into another rectangle in such a way that the new width becomes the excess of the length over the width, $x - y$, and the new length becomes the semi-perimeter of the original rectangle increased by one, $x + y + 1$. In how many ways can this be done if the area of the new rectangle should be equal to $a$ (square units)?

This geometric brainteaser can be represented in an algebraic form through the two-variable Diophantine equation 6 with an integer parameter $a$.

$$(x - y)(x + y + 1) = a \tag{6}$$

In comparison with definition 2 where the constraint on the value of $x_n$ was introduced directly, equation 6 does not provide one with direct information about ranges within which the variables $x$ and $y$ and parameter $a$ vary. Instead, such information should be extracted from equation 6 using mathematical reasoning skills. As these variables are linked to each other through an equation, knowing the range for one variable should yield the range for another variable. By the same token, the specific form of equation 6 can reveal hidden information about parameter $a$.

We begin with introducing

**Constraint for Parameter $a$**. *Equation 6 has no solutions when $a$ is an odd number.*

**Proof**. As $x-y$ and $x+y$ are of the same parity, the factors $x-y$ and $x+y+1$ are of different parity. Therefore, the products in the left-hand side of equation 6 can only be an even number and so is parameter $a$.

**Remark.** In constructing the above argument (called proof) one uses the basic skill of making a parity check for a sum and difference of two whole numbers at a more advanced level in order to make a parity check for a product of two whole numbers. To support the development of advanced skills in making a formal argument, one can use the basic skill of using numerical evidence as heuristic and demonstrative method [8]. That is, one can first discover through numerical evidence that parameter $a$ may not be an odd number and support this discovery through pure reasoning afterwards. This can also include a skill of drawing geometric figures, which enable one to connect symmetrical and pairing properties of objects. Figure 4 shows how drawing capabilities of a spreadsheet can be used in support of the brainteaser and enabling an informal deduction of the even-numbered property of parameter $a$ in equation 6.

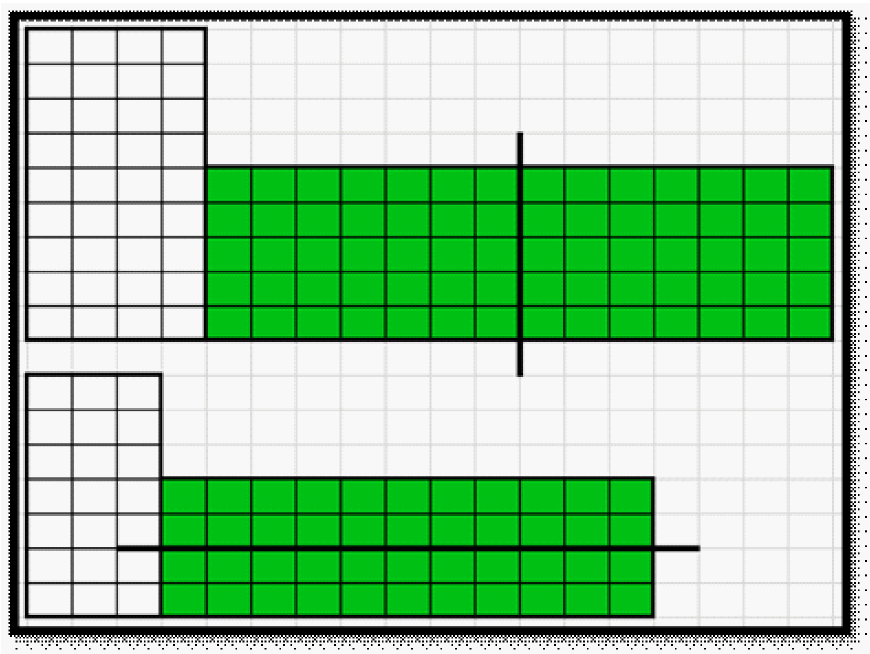


Figure 4: Recognizing symmetry through pairing objects.

## 4.2 The skill of using the law of excluded middle

**The first constraint for variable** $x$. *The variable* $x$ *in equation 6 satisfies the inequality*

$$x \leq \frac{a}{2} \tag{7}$$

**Proof.** Assuming the contrary, that is, assuming that $x > \frac{a}{2}$ yields $a = (x-y)(x+y+1) = x^2 - y^2 + x - y > \frac{a^2}{4} - y^2 + \frac{a}{2} - y$ whence

$$y^2 + y + \frac{a}{2} - \frac{a^2}{4} > 0 \tag{8}$$

A possible solution to inequality 8 is given by the inequality $y > \frac{-1+\sqrt{1+a^2-2a}}{2} = \frac{-1+a-1}{2} = \frac{a}{2} - 1$. In other words, the assumption $x > \frac{a}{2}$ yields $y > \frac{a}{2} - 1$ whence $x + y + 1 > a$. The last inequality, however, contradicts equation 6 because none of the whole number factors may be greater than their product.

**Remark 1**. The above constraint on the value of $x$ was developed using a specific mathematical skill known as proof by contradiction. By trying to extend a basic skill of using constraints in the form of inequalities, one develops an advanced skill of using the so-called law of excluded middle – since a statement cannot be false, it must then be true. The question remains as to how can one come up with inequality 7 and not with a different inequality? The ability of making a mathematically meaningful statement, although it could be ascribed to mathematical intuition, which includes combining different ideas and selecting the useful one in unconscious way [15], requires special training in *doing* mathematics. Put another way, the development of mathematical skills occurs within one's zone of proximal development with the assistance of a more knowledgeable other [25]. In the age of technology, such assistance can be effectively provided through a numerical approach. Indeed, by using a spreadsheet as a two-dimensional modeling tool, one can discover that whatever the value of parameter $a$, no solution to equation 6 includes an $x$-value that is greater than $\frac{a}{2}$. In that way, a novice practice in spreadsheet modeling can produce the initial insight that leads to the proof and, ultimately, turns into an expert practice in using a spreadsheet. In much the same way, spreadsheet modeling can support the guessing of other constraints for the variables in equation 6 discussed below. More details on the role of a spreadsheet in generating conjectures and motivating proof can be found in [1], [3].

**The first constraint for variable** $y$. *The variable* $y$ *in equation 6 satisfies the inequality*

$$y \leq \frac{a}{2} - 1.$$

**Proof.** Assuming $y > \frac{a}{2} - 1$ (that is, making an assumption contrary to what has to be proved) yields

$$a = x^2 - y^2 + x - y < \frac{a^2}{4} + \frac{a}{2} - \left(\frac{a}{2} - 1\right)^2 - \left(\frac{a}{2} - 1\right) = a.$$

That is, we arrived to a false inequality, $a < a$. Therefore, as the inequality $y \le \frac{a}{2} - 1$ cannot be false it must then be true.

**The second constraint for variable** $x$. *The largest value of $x$ smaller than $\frac{a}{2}$ that satisfies equation 6 cannot be greater than $\frac{a}{4} + \frac{1}{2}$.*

**Proof**. Let the factors in the left-hand side of equation 6 be $x-y = 2$ and $x+y+1 = \frac{a}{2}$. Then their sum $2x+1 = \frac{a}{2}+1$ whence $x = \frac{a}{4} + \frac{1}{2}$ and $y = \frac{a}{4} - \frac{3}{2}$. Therefore, if $a$ is an even number, then $\left(\frac{a}{2},\ \frac{a}{2} - 1\right)$ is a solution to equation 6; otherwise, the largest value of $x$ smaller than $\frac{a}{2}$ that satisfies equation 6 is smaller than $\frac{a}{4} + \frac{1}{2}$.

**The second constraint for variable** $y$**.** *The largest value of the variable $y$ smaller than $\frac{a}{2} - 1$ that satisfies equation 6 is not greater than $\frac{a}{4} - \frac{3}{2}$.*

**Proof**. Assuming $x \le \frac{a}{4} + \frac{1}{2}$ yields

$$a = x^2 - y^2 + x - y \le \left(\frac{a}{4} + \frac{1}{2}\right)^2 - y^2 + \frac{a}{4} + \frac{1}{2} - y = \frac{a^2}{16} + \frac{a}{2} + \frac{3}{4} - y^2 - y$$

whence $y^2 + y - \frac{a^2-8a+12}{16} \le 0$. Solving the last inequality yields $y \le \frac{a}{4} - \frac{3}{2}$.

**Remark 2**. Nowadays, the use of computer algebra systems, like *Maple*, makes it possible to carry out the above proofs using symbolic computations. Obviously, using *Maple* to assist in formal demonstration requires the development of new skills that are far from basic ones. It appears, however, that the advent of computer algebra systems in the modern educational environment would not cause skills in rigorous mathematical proof to die out (as, for example, the skill of extracting square root taught in the schools until recently) but rather, proof assistant technology would be used by mathematicians as a means "to put the correctness of their proofs beyond reasonable doubt" [16, p1405].

### 4.3 Tables and graphs as means of representation and analysis

Mathematical reasoning skills used above made it possible to establish ranges for the variables $x$ and $y$ as $[1,\ \frac{a}{4} + \frac{1}{2}]$ and $[1,\ \frac{a}{4} - \frac{3}{2}]$, respectively, where $a$ is an even number. One can use a spreadsheet to construct a function that relates even values of parameter $a$ to the number of solutions of equation 6. In particular, the case $2 \le a \le 80$ requires the range $[1, 20]$ for the variable $x$ and $[1, 19]$ for the variable $y$, keeping in mind that equation 6 has an obvious solution $x = \frac{a}{2},\ y = \frac{a}{2} - 1$. Figure 5 shows a fragment of the spreadsheet that models equation 6 using the ranges that do not include the obvious solution. In particular, the spreadsheet pictured in Figure 5 does not include the solution $(20, 19)$. Figure 6 shows both table and graphical representations of the function $NR(a)$ that relates $a$ to the number of rectangles that satisfy equation 6. One can appreciate the significance of elimination of odd values of parameter $a$ from modeling that enables one to generate data within a sufficiently large range. For more details on the use of inequalities as tools in spreadsheet-based computing applications see [1].

To conclude this section note that another important skill that develops through numerical evidence is conjecturing based on the interpretation of modeling data. For example, one can see that when $a$ is a power of two, equation 6 has the trivial solution only. On the other hand, other values of $a$ such as $6, 12, 20, 56$ provide only trivial solutions as well. What is special about these values? By the same token, when $a = 54, 66$, and 78, equation 6 has four solutions—the largest number in comparison with other values of $a$. What is the smallest value of $a$ for which equation 6 has five solutions? The development of skills in conjecturing continues as one moves along the $\mathrm{K}^2\mathrm{A}^2$ staircase.

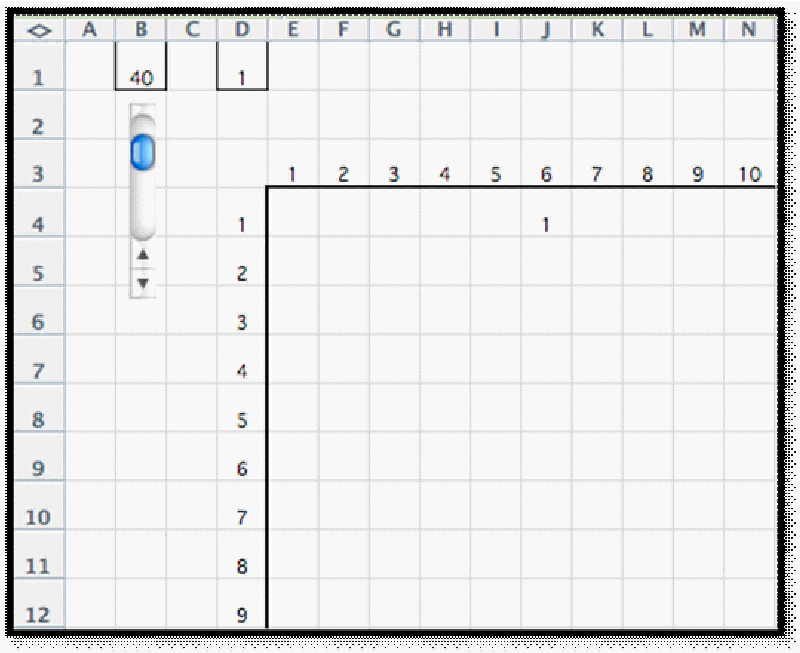
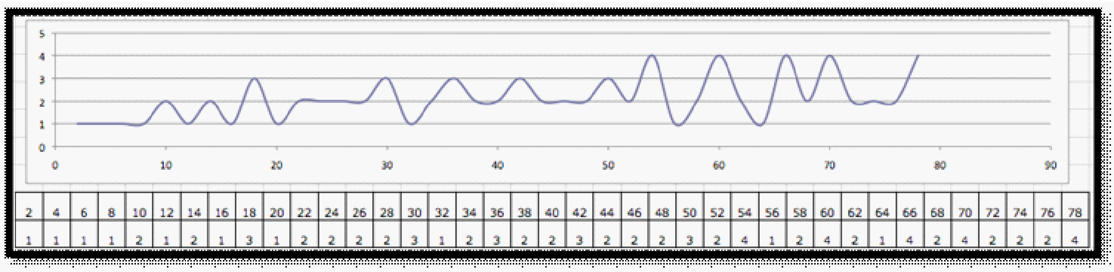



Figure 5: Modeling equation 6 without counting an obvious solution (the case $a = 40$).




| | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 2 | 4 | 6 | 8 | 10 | 12 | 14 | 16 | 18 | 20 | 22 | 24 | 26 | 28 | 30 | 32 | 34 | 36 | 38 | 40 | 42 | 44 | 46 | 48 | 50 | 52 | 54 | 56 | 58 | 60 | 62 | 64 | 66 | 68 | 70 | 72 | 74 | 76 | 78 |
| 1 | 1 | 1 | 1 | 2 | 1 | 2 | 1 | 3 | 1 | 2 | 2 | 2 | 2 | 3 | 1 | 2 | 3 | 2 | 2 | 3 | 2 | 2 | 2 | 3 | 2 | 4 | 1 | 2 | 4 | 2 | 1 | 4 | 2 | 4 | 2 | 2 | 2 | 4 |

Figure 6: Table and graphic representation of the function $NR(a)$.

## 5 Illustration 3: reaching the limitations of a single tool

Another important skill in the professional use of technology is the ability of the appropriate use of multiple software tools. There are many problems in the STEM disciplines where the use of a spreadsheet alone is inadequate. An advanced skill of using a spreadsheet can be developed through experiencing its limitation and practicing its integration with other computer applications. In particular, in engineering sciences many theoretical models do not allow for an exact solution and can only be approached from a computational direction.

As an example, consider the following two simultaneous equations with four parameters that describe thermodynamic processes occurring in solid solutions of certain semiconductor systems [21], [22].

$$\ln\left(\frac{x}{y}\right) = V_1(1-x^2) - V_2(1-y^2) - a, \quad \ln\left(\frac{1-y}{1-x}\right) = V_1x^2 - V_2y^2 - b \tag{9}$$

A complex computational problem is to find pairs of $(x, y)$ satisfying equations 9 when the four parameters are appropriately chosen. The knowledge of such pairs makes it possible to construct the so-called diagrams of the complete mutual solubility of solutions in both phases [21].

The importance of the development of skills of physicochemical calculations using graphical methods has been emphasized for more than half a century [14]. However, in the past, such calculations were extremely time consuming and intellectually challenging. Nowadays, modern tools of technology can significantly enhance the construction of graphs of complex equations thereby turning the calculation skills previously considered advanced into basic skills but at a higher cognitive level. Moreover, by using technology in complex scientific calculations, one can develop the ability of transferring skills from one professional context to another. It is through such a meaningful practice that one type of skills transforms into another and vice versa.

In the context of equations 9, once again the problem of determining ranges for the variables $x$ and $y$ can be formulated. However, unlike two-variable Diophantine equation 6, which allowed for mathematical reasoning skills to be used in defining efficient ranges for the variables involved, such ranges in the present case can only be determined graphically. However, a spreadsheet does not allow for the construction of graphs from two-variable equations.

A possible approach to overcome this limitation of a spreadsheet is to use it jointly with software that allows for the construction of graphs from any two-variable equation depending on multiple parameters. The Graphing Calculator 3.5 ($GC$) produced by Pacific Tech (`http://www.nucalc.com`) is one such tool. As shown in Figure 7, the $GC$ made it possible to construct graphs of equations 9 for those values of parameters $a$, $b$, $V_1$, and $V_2$ that provide a mutual solution. Due to the continuity of the functions involved, one can assume that is a certain neighborhood of the point of intersection of the two graphs multiple cases of such intersection can be found. Those cases now can be picked up by a spreadsheet. Figure 8 shows a cigar-like set of points $(x, y)$ satisfying equations 9 for certain values of the four parameters. This example shows that what one tool cannot do, the other tool can do and vice versa. Once again, the ability to use multiple software tools in scientific investigations can first be characterized as an advanced skill that, after being applied to different professional contexts, gives rise to new problems that transform the skill into a basic one. Although a spreadsheet can be used as a tool kit, it does have certain limitations. This reinforces the importance of its skilful use jointly with other technologies.

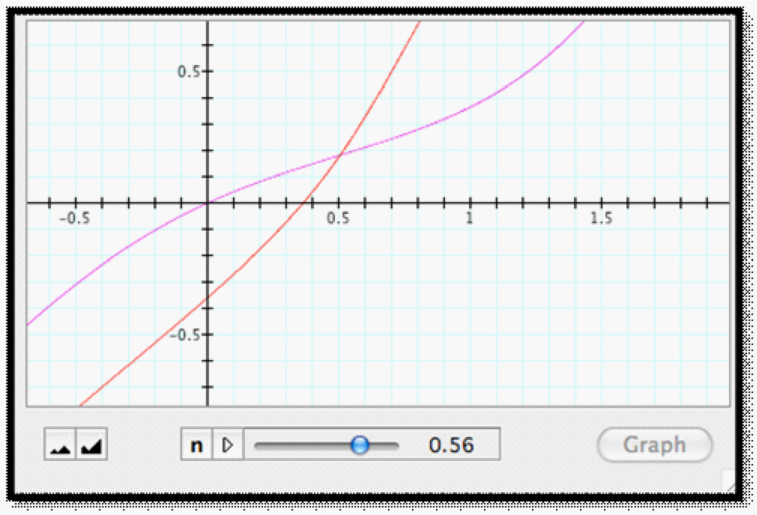
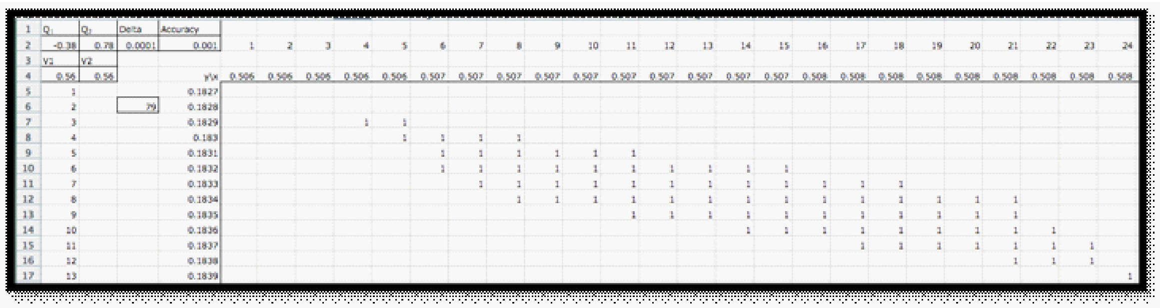



Figure 7: The GC-based determination of ranges for the variables $x$ and $y$.



| | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | $Q_1$ | $Q_2$ | Delta | Accuracy | | | | | | | | | | | | | | | | | | | | | | | | |
| 2 | -0.38 | 0.78 | 0.0001 | 0.001 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 |
| 3 | V1 | V2 | | | | | | | | | | | | | | | | | | | | | | | | | | |
| 4 | 0.56 | 0.56 | | y\x | 0.506 | 0.506 | 0.506 | 0.506 | 0.506 | 0.507 | 0.507 | 0.507 | 0.507 | 0.507 | 0.507 | 0.507 | 0.507 | 0.507 | 0.507 | 0.508 | 0.508 | 0.508 | 0.508 | 0.508 | 0.508 | 0.508 | 0.508 | 0.508 |
| 5 | 1 | | | 0.1827 | | | | | | | | | | | | | | | | | | | | | | | | |
| 6 | 2 | | 79 | 0.1828 | | | | | | | | | | | | | | | | | | | | | | | | |
| 7 | 3 | | | 0.1829 | | | | 1 | 1 | | | | | | | | | | | | | | | | | | | |
| 8 | 4 | | | 0.183 | | | | | 1 | 1 | 1 | 1 | | | | | | | | | | | | | | | | |
| 9 | 5 | | | 0.1831 | | | | | | 1 | 1 | 1 | 1 | 1 | 1 | | | | | | | | | | | | | |
| 10 | 6 | | | 0.1832 | | | | | | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | | | | | | | | | |
| 11 | 7 | | | 0.1833 | | | | | | | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | | | | | | |
| 12 | 8 | | | 0.1834 | | | | | | | | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | | | |
| 13 | 9 | | | 0.1835 | | | | | | | | | | | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | | | |
| 14 | 10 | | | 0.1836 | | | | | | | | | | | | | | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | | |
| 15 | 11 | | | 0.1837 | | | | | | | | | | | | | | | | | 1 | 1 | 1 | 1 | 1 | 1 | 1 | |
| 16 | 12 | | | 0.1838 | | | | | | | | | | | | | | | | | | | | | 1 | 1 | 1 | |
| 17 | 13 | | | 0.1839 | | | | | | | | | | | | | | | | | | | | | | | | 1 |

Figure 8: Spreadsheet modeling of equations (9) within a specified domain in the $x - y$ plane.

## 6 Concluding remarks

This paper has demonstrated how a spreadsheet can be used as a medium for the development of three types of skills—basic, professional, and advanced—required for the STEM workforce of the future. It showed how technology, in general, and a spreadsheet, in particular, can support the introduction of mathematical concepts through using basic skills in professionally-oriented computing applications. The authors argue that in order to promote interest in the STEM disciplines and develop appropriate skills at the college level and beyond, a spreadsheet can be recommended as an educational tool that provides one with experience in moving from novice practice to expert practice in solving a variety of problems through modeling.

By using a spreadsheet, advanced skills that are connected to one's conceptual understanding of a task in question can emerge from basic skills. Moreover, through repeated applications, skills that are considered advanced for a novice practice have the potential to become basic skills of an expert practice in spreadsheet modeling. The possibility of using the software as a computationally efficient and visually enhanced learning environment makes it possible to study real-life situations of increasingly complex nature and compare them through analyzing modeling data. Various facilities of a spreadsheet, including recurrent counting, interactive construction of graphs of function not defined by a formula, drawing integer-sided rectangles and other types of polygons, using mathematical definitions to format numerical information, make it possible for learners to see a manifold of ideas in a clear and natural way. Using these features of a spreadsheet in the preparation teachers, mathematicians, scientists, engineers and other professionals can significantly improve the development of skills that can be transferred across different disciplines and contexts.

## References

[1] Abramovich, S. (2005). Inequalities and spreadsheet modeling. *Spreadsheets in Education* [Online journal], **2**(1): 1-22. Available at: http://epublications.bond.edu.au/ejsie/vol2/iss1/1.

[2] Abramovich, S., Nikitina, G., and Romanenko, V. (2002). Developing practical competence of future engineers within a theory-oriented curriculum at the tertiary level. *Herald of Education and Science Development of Russian Academy of Natural Sciences* (special issue in English), **4**: 24-30.

[3] Abramovich, S., and Sugden, S. (2004). Spreadsheet conditional formatting: an untapped resource for mathematics education. *Spreadsheets in Education* [Online journal], **1**(2): 85-105. Available at: http://epublications.bond.edu.au/ejsie/vol1/iss2/3.

[4] Abramovich, S. and Sugden, S. (2008). Diophantine equations as a context for technology-enhanced training in conjecturing and proving. *Problems, Resources, and Issues in Mathematics Undergraduate Studies (PRIMUS)*, **18**(3): 257-275.

[5] Aleksandrov, A. D. (1965). A general view of mathematics. In A. D. Aleksandrov, A. N. Kolmogorov, and M. A. Lavrent'ev (Eds), *Mathematics: Its Content, Methods and Meaning*, pp. 1-64 Cambridge, MA: MIT Press

[6] Arganbright, D. E. (1985). *Mathematical Applications of Electronic Spreadsheets.* New York: MacGraw-Hill.

[7] Berg, E. E. (1970). *L. S. Vygotsky's Theory of the Social and Historical Origins of Consciousness* (Dissertation). Ann Arbor, MI: University Microfilms, Inc.

[8] Beth, E. W. (1966). Strict demonstration and heuristic procedures. In E. W. Beth and J. Piaget, *Mathematical Epistemology and Psychology*, pp. 86-100. Dordrecht, The Netherlands: Reidel.

[9] Bruner, J. S. (1966). *Toward a Theory of Instruction.* Cambridge, MA: Harvard University Press.

[10] Conference Board of the Mathematical Sciences. (2001). *The Mathematical Education of Teachers*. Washington, D.C. The Mathematical Association of America.

[11] Davis, P. J., Hersh, R., and Marchisotto, E. A. (1995). *The Mathematical Experience*. Boston: Birkhäuser.

[12] Filby, G. (ed.). (1998). *Spreadsheets in Science and Engineering.* Berlin: Springer-Verlag.

[13] Freudenthal, H. (1983). *Didactical Phenomenology of Mathematical Structures*. Dordrecht: Reidel.

[14] Guggenheim, E. A., and Prue, J. E. (1956). *Physicochemical Calculations.* North Holland Publishing Company: Amsterdam.

[15] Hadamar, J. (1954). *The Psychology of Invention in the Mathematical Field.* New York: Dover Publications.

[16] Harrison, J. (2008). Formal proof—theory and practice. *Notices of the American Mathematical Society*, **55**(11): 1395-1406.

[17] Howard, R. W. (1987). *Concepts and Schemata*. Philadelphia, PA: Cassell.

[18] Katehi, L., Pearson, G., and Feder, M. (2009). *Engineering in K-12 Education.* Washington, DC: The National Academies Press.

[19] Kemp, I. J., and Seagraves, L. (1995). Transferable skills – can higher education deliver? *Studies in Higher Education*, **20**(3): 315-328.

[20] Poyton, T. A. (2004). Computer literacy across the lifespan: a review with implications for educators. *Computers in Human Behavior*, **21**(7): 861-872.

[21] Romanenko, V. N., and Ivanov-Omsky, V. I. (1959) On the thermodynamic of the solid solutions of certain semi-conductor systems. *Transactions of Soviet Academy of Sciences, Technical physics series*, **129**(3): 353-355.

[22] Schilz, J., and Romanenko, V. N. (1995). Review: Bulk growth of silicon-germanium solid solutions. *Journal of Materials Science: Materials in Electronics*, **6**: 265-279.

[23] Sheppard, S. D., Macatangay, K., Colby, A., and Sullivan, W. M. (2009). *Educating Engineers*. San Francisco: Jossey-Bass.

[24] Shulman, L. (1986). Those who understand: knowledge growth in teaching. *Educational Researcher*, **15** (2): 4-14.

[25] Vygotsky, L. S. (1978). *Mind in Society*. Cambridge, MA: Harvard University Press.

[26] Vygotsky, L. S. (1986). *Thought and Language*. Cambridge, MA: MIT Press.

[27] Vygotsky, L. S. (1987). Thinking and speech. In R. W Rieber, and A. S. Carton (Eds), *The Collected Works of L. S. Vygotsky*, vol. 1 (pp. 39 – 285). New York: Plenum Press.

[28] Wankat, P., and Oreovicz, F. S. (1993). *Teaching Engineering*. New York: McGraw-Hill.