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A matter of interpretation: bargaining over ambiguous contracts

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Abstract

We present a formal treatment of contracting in the face of ambiguity. The central idea is that boundedly rational individuals will not always interpret the same situation in the same way. More specifically, even with well defined contracts, the precise actions to be taken by each party to the contract might be disputable. Taking this potential for dispute into account, we analyze the effects of ambiguity on contracting. We find that risk averse agents will engage in ambiguous contracts for risk sharing reasons. In addition to the risk sharing motivations for contracting in the presence of ambiguity, we find that agents may contract in order to reduce the downside effects of non-cooperative opportunism arising from ambiguity.

JEL Classification: D80, D82

Key words: ambiguity, bounded rationality, incomplete contracts

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1 Introduction

Language is a matter of interpretation, and interpretations will differ. This fact is of fundamental importance in the construction of contracts, which are written or verbal agreements that the parties act in particular ways under particular conditions. For any contract to be successfully implemented, the parties must agree on whether the relevant conditions apply. A contract that is ambiguous, in the sense that parties may differ in their interpretation of the conditions that apply, and therefore of the actions that are required, will lead to disputes and, ultimately, litigation.

To avoid disputes, parties to a contract may seek to avoid ambiguous terms, even when the resulting contract is incomplete, in the sense that opportunities for risk-sharing or productive cooperation are foregone. For example, parties may adopt a standard contract, in which the terms are well-defined as a result of established precedents, even if a variation on the standard contract could potentially yield a Pareto improvement.

Although the problems with ambiguous contracts have been much discussed in legal literature, the central point that ambiguous contractual terms can lead to incomplete contracts has received relatively little attention from economists. This is because contracts are typically modelled as state-contingent acts, with incompleteness arising from the fact that some states may be non-contractible or from state-contingent preferences that are ambiguous, in the technical sense that there exists no well-defined probability distribution over the state space. The language in which contracts are written is either not specified or derived from the state space.

The idea that incompleteness in contracts arises from an inability to specify and contract on the state space is not new; it has been a standard argument at least since Williamson (1975, 1985) drew attention to the importance of transactions costs in determining contractual structures. These transactions costs are typically imputed to incompleteness of the state space. However, as Maskin and Tirole (1999) observe, incompleteness of the state space is not, in itself, sufficient to preclude the achievement of the first best contract. Provided that the optimal contract does not depend on welfare-irrelevant distinctions between states, Maskin and Tirole show that an optimal contract may be achieved that depends only on welfare outcomes and not on knowledge of the physical state space.

Segal (1999) has argued that in some complex environments, distinctions between complete and incomplete contracts might become trivial. Bernheim and Whinston (1998) showed that incomplete contracts might be chosen by agents who face strategic ambiguity. Spier (1992) has argued that incomplete contracts might be chosen as signalling devices. Mukerji (1998) and Mukerji and Talon (2001) discussed incomplete contracts in the presence of the decision-theoretic concept of 'ambiguity', which refers to a situation in which an agent's preferences cannot be rationalized by a specific probability distribution over a commonly known state space.

In this paper, we adopt an alternative approach, in which the language in which contractual terms are specified is taken as primitive. A contract is simply a set of conditional actions, built up using an '**if** t **then** a **else** a'' where t is a contractual term (or test) and a and a' are actions. Following the constructive approach to decision theory developed by Blume, Easley and Halpern (2006), we show how any preference relation over contracts written in the given language gives rise, in a natural fashion, to a state space with an associated state-dependent utility function.

We then consider contracts between two parties, using the same contractual language but with possibly different interpretations of the tests specified in the contract. Tests that are subject to the possibility of disagreement are described as ambiguous, while those for which there is no possibility of disagreement are unambiguous. Even though we assume it is commonly known by the parties whether a test is ambiguous or not, in situations where the parties disagree over the outcome of an ambiguous test, disputes may still arise.

It is natural, in this context, for parties to be averse to ambiguity since disputes are costly and may be resolved with an interpretation of the contract less favorable than that which would have obtained in the absence of ambiguity. We suggest a bound for the cost of disputes and show that it naturally gives rise to preferences that may be represented by an ε-contamination model, a special case of the multiple priors model of Gilboa and Schmeidler (1989), which has been used to represent ambiguous state-contingent preferences. Thus, our approach gives rise to a natural connection between aversion to linguistic ambiguity (the sense in which the term 'ambiguity' is normally found in ordinary usage) and state-contingent ambiguity (the sense in which the term is commonly used in decision theory).

Given these preferences, we show that a two-agent bargaining process of the type modelled by Rubinstein, Safra and Thomson (1992) leads to a unique *ordinal* Nash bargaining outcome. For the case of a risk-sharing contract, the equilibrium involves a trade-off between risk and ambiguity. A finer contractual specification (in a sense which can be made precise) increases the gains from risk sharing when the contract is implemented successfully, but also increases the ambiguity of the contract and creates more possibilities for dispute. In this context, we find that risk aversion makes agents more likely to engage in contracts involving ambiguous terms and discuss the trade off between risk aversion and willingness to contract in the face of ambiguity.

In a game setting, the ambiguity could also be about the actions of an opponent. In this context, contracts can serve to reduce the uncertainty one has about the other's actions. We analyze the effects of ambiguity on contracting for ambiguity averse agents playing a coordination game.

The paper is organized as follows. We begin with an illustrative example. In section 3, we set up the formal language in which contracts are specified, and show how state-dependent "ambiguity-free" preferences over contracts may be axiomatized. Next, in Section 4, we develop the concept of contractual ambiguity, and derive preferences over ambiguous contracts. In Section 5 we formulate and solve the associated bargaining problem. In Section 6, we consider a coordination problem where contracting helps reduce the downside in an ambiguous setting. This example shows how the benefits from contracting in the face of ambiguity extend beyond those from pure risk sharing. In Section 7 we discuss the implications of our analysis and its relationship to the existing literature on incomplete contracts and bounded rationality.

2 Illustrative example

In informal discussions of ambiguous contracts, it is common to refer to 'gray areas'. Some contracts, or contingencies specified in contracts, are seen as having gray areas, thereby giving rise to possibilities of disagreement and dispute, while others are seen as relatively clear-cut and unambiguous.

We develop these ideas in an example, specified as follows.[1] Suppose two individuals *Row* (Rowena) and *Col* (Colin) are contemplating entering into a risk-sharing contract. They will draw a card from a pack. It may be red at both ends, black at both ends, white at both ends, or it may be black at one end and white at the other. If both ends are red (black or white) the card is deemed 'red' ('black' or 'white', respectively). The cards are vertically oriented so that the card 'black at top, white at bottom' is different from 'white at top, black at bottom'. Hence, from the viewpoint of an unboundedly rational observer there are five possible states of the world, one for each card.

Each player sees the world as red, black or white. However, *Row* always observes the top half of the card, while *Col* always observes the bottom half. Thus, if the card is black at the top and white at the bottom, *Row* will construe the card is black, while *Col* will construe it as white. The underlying state space and the two individuals' partitions of the black–white spectrum are summarized in the following table, where X denotes a pair of observations that is inconsistent with the problem description and therefore does not correspond to a state:

		Col's observation		
	Card drawn is:	*red (at bottom)*	*black (at bottom)*	*white (at bottom)*
	red (at top)	red,red	X	X
Row's observation	*black (at top)*	X	black, black	black,white
	white (at top)	X	white,black	white,white

Suppose the state-contingent endowments of the two individuals are given in the following bi-

[1] We are indebted to Bob Brito for this suggestion.

matrix,

		Col's endowment		
	Card drawn is:	*red at bottom*	*black at bottom*	*white at bottom*
	red at top	1 3	X	X
Row's endowment	*black at top*	X	3 1	2 1
	white at top	X	3 2	2 2

Each individual faces a single source of uncertainty that is measurable with respect to his own partition of the state space. We assume that both players are risk-averse and view the three elements of their respective partitions as 'exchangeable' (Chew and Sagi 2006).[2] Hence both parties would prefer the non state-contingent allocation $(2, 2)$ in every state. So, ignoring (for the moment) any possibility of future disagreement and dispute, both would find it attractive to sign a risk-sharing contract inducing the following contingent transfer from *Col* to *Row* :

$$f_c = \begin{cases} -1 & \text{if the card drawn is `red'} \\ 1 & \text{if the card drawn is `black'} \\ 0 & \text{if the card drawn is `white'.} \end{cases}$$

In the formal framework developed below, if such a contract were signed, the presumption is that each party will assess which contingency has obtained according to her or his own semantics. For *Row*, this entails assessing that 'the card drawn is black' is true when she makes the observation the card drawn 'is black (at the top)', while for *Col*, this entails assessing that 'the card drawn is black' is true when he observes the card drawn 'is black (at the bottom)'.

[2] In this context, 'exchangeable' is equivalent to each individual being indifferent between betting on any element of his or her partition.

The card that is black at the top and white at the bottom creates a possibility for disagreement since *Row* will interpret this as 'black', and so believe that she is entitled to receive a payment. *Col* will in the same situation interpret this as 'white', so he will expect no payment is required. Hence, a disagreement will ensue.

Boundedly rational players, in this setup, are unable (in the absence of some increase in effort) to formulate a state description sufficiently refined to encompass this possibility, allowing the contract to specify a resolution. However, they may nonetheless be aware that disputes are possible. Depending on the weight they place on this possibility, they may choose a contract which offers only partial hedging, or even no contract at all. This corresponds closely to the risk–uncertainty distinction of Knight (1921) whose main concern was with uncertainties that could not be hedged through market contracts such as insurance, and therefore reduced to manageable risk. Uncertainty of this kind was central to Knight's idea of entrepreneurship.

While we do not formally model the awareness and knowledge of the players using epistemic logic as in, for example. Fagin et al. (1995), some comment on their presumed awareness and knowledge is necessary. The awareness of the players includes a number of elements. First, each player is aware of their own state-contingent description of the world and of the information available to them. Second, given the description above, each is aware that the other may not have access to their model of the world. In this example, *Row* and *Col* are both aware that each of them is aware of the statements 'the card is red' and 'the card is black'. However, each is also conscious that their model and the model of the other individual may be incomplete. In particular there may exist other details about the world of which neither is currently aware, that lead to different interpretations by the two about the semantic content of those statements for the two players. But, as noted in the introduction, we further assume that it is common knowledge whether a test is ambiguous or not.

Thus, the central feature of the example is that players are boundedly rational, but nonetheless sophisticated enough to reason about their own bounded rationality and that of others. This is consistent with the observation of Maskin and Tirole (1999, p 106) that the central problem in contracting is not incompleteness *per se* but bounded rationality:

> "if we are to explain 'simple institutions' such as property rights, authority (or more generally decision processes) short-term contracts and so forth a theory of bounded rationality is certainly an important, perhaps ultimately an essential ingredient."

The central concern of this paper is to develop a model of contracting between parties whose bounded rationality is embodied in the ambiguity of the language they use to describe the world. To this end, it is crucial to relate propositional or syntactic descriptions of the world to an underlying state space when the parties involved are boundedly rational. We will follow the constructive decision theory approach of Blume, Easley and Halpern (2006) in which the propositional representation is taken as primitive, along with the set of actions on which contingent contracts can be written.

3 Formal languages, contracts and "ambiguity-free" preferences

We consider two parties $i = 1, 2$, and following the approach of Blume et al (2006), we assume that both players have access to a non-empty set of primitive test propositions $T_0 = \{t_1, ..., t_K\}$ and a set of actions A_0. Let T denote the closure of T_0 under conjunction ($\wedge$) and negation ($\neg$).

A typical action $a \in A_0$ might be 'player i performs service z for player j in return for consideration w'. Formally, we take A_0 to be a compact and convex subset of a separable metric space.

We are interested in the set of contracts C, which are constructed inductively from the set of actions A_0 and the set of tests T_0 by taking the closure under the 'if-then-else' construction. That is, we take each a in A_0 to be a contract, and then we require, for any pair of contracts c and c' and any test t in T_0, that the program '**if** t **then** c **else** c'' should be a contract as well. Hence, for any pair of contracts c and c' and any test t in T, the program '**if** t **then** c **else** c'' is also a contract in C. This contract requires the parties to follow the course of action as determined by contract c if test t is satisfied and follow the course of action as determined by contract c' otherwise.

Tests and contracts are simply strings of symbols with no inherent semantic content. The semantics will be derived from preferences of each individual rather than being given in advance. More precisely, we derive, for each player, a state space S^i. Although Blume et al (2006) allow for the non-uniqueness of a state space, we adopt their canonical state space and set $S^i = S = 2^{T_0}$ for $i = 1, 2$ and refer to it as S hereafter.

An element $s = (s_1, ..., s_K) \in S$ is a vector of zeros and ones and we use s_i to denote the i'th component of s. A test interpretation is a function $\pi : T \to 2^S$, where $\pi(t)$ is the set of states in which the test t is true. Notice that the state space induces a test interpretation constructed as follows. For each t_i in T_0, $\pi(t_i) = \{s \in S : s_i = 1\}$. The test interpretation is then inductively extended to tests in T by the rule: for any $t, t' \in T$, $\pi(t \wedge t') = \pi(t) \cap \pi(t)$, and $\pi(\neg t) = S - \pi(t)$.

Each state $s \in S$ can be identified with a test $t(s) = t_1(s) \wedge ... \wedge t_K(s) \in T$ where the test is defined as follows. For each $s \in S$ and each $i = 1, ..., K$ let:

$$t_i(s) = \begin{cases} t_i \text{ if } s_i = 1; \\ \neg t_i \text{ if } s_i = 0. \end{cases}$$

By construction $\pi(t(s)) = \{s\}$ meaning the test $t(s)$ is satisfied only at the state s.

For any $a \in A_0$, f_a is the unconditional act $f_a(s) = a$ for all $s \in S$. Fix a pair of contracts c and c' in C with associated state-contingent actions f_c and $f_{c'}$. Then for any test t in T, the state-contingent action associated with the contract $c'' =$ '**if** t **then** c **else** c'' is given by $f_{c''}(s) = f_c(s)$ if $s \in \pi(t)$, and $f_{c''}(s) = f_{c'}(s)$ if $s \notin \pi(t)$. Hence it follows from the inductive construction of the set of contracts above that for each c in C, there is an associated 'state-contingent' action $f_c : S \to A_0$.

Consider now the individuals' 'ambiguity-free' preferences $\succsim^i$, $i = 1, 2$, defined over C. These should be interpreted as the players' preferences over contracts, in the absence of any consideration of possible disputes. That is, these are the preferences each player has under the assumption that the other party has the same understanding of the contractual terms, and that the contract is implemented according to this shared understanding. In the next section we consider the possibility of a dispute arising from different interpretations of 'ambiguous' tests.

We assume these preferences satisfy the following three axioms. The first is the standard ordering axiom. The second requires that any two contracts that induce the same act over actions must come from the same indifference class. This seems natural in a setting in which we assume the agent understands the language in which contracts are written and the logical implications of its terms and attendant requirements. The third axiom is the analog of Savage's *sure-thing principle.*

Ordering Axiom The relation $\succsim^i$ is complete and transitive.

Act-equivalence Axiom For any pair of contracts, c and c' in C, if $f_c = f_{c'}$ then $c \sim c'$.

***Sure-thing Principle*:** For any four contracts c, c', c'' and c''' in C, and any test t in T,

$$\begin{aligned} & \textbf{if } t \textbf{ then } c \textbf{ else } c'' \succsim^i \textbf{if } t \textbf{ then } c' \textbf{ else } c'' \\ \Rightarrow \quad & \textbf{if } t \textbf{ then } c \textbf{ else } c''' \succsim^i \textbf{if } t \textbf{ then } c' \textbf{ else } c'''. \end{aligned}$$

Our fourth axiom is a continuity assumption to ensure that a numerical representation of preferences exists. Before stating it, we need to define what it means for a sequence of contracts to converge to a limit. We do this inductively. First, we define the notion of convergence for constant acts directly from the notion of convergence of actions in the set A_0, and then we extend it inductively to all contracts via the 'if..then..else' construction.

Definition 1 (Convergence of Sequences of Contracts) *(a) (base case) The (countably infinite) sequence of constant acts $\langle a_n \rangle$* converges *to the constant act $\bar{a}$, if the corresponding sequence of actions converge to the corresponding action, that is,* $\lim_{n\to\infty} a_n = \bar{a}$.
*(b) (inductive extension) For any sequence of tests $\langle t_n \rangle$ and any pair of sequences of contracts $\langle c_n \rangle$ and $\langle c'_n \rangle$, the sequence of contracts $\langle c''_n \rangle$, where $c''_n =$ '**if** t_n **then** c_n **else** c'_n',* converges *to $\bar{c}'' =$ '**if** $\bar{t}$ **then** $\bar{c}$ **else** $\bar{c}'$', if $\langle c_n \rangle$ and $\langle c'_n \rangle$ converge to $\bar{c}$ and $\bar{c}'$, respectively, and there exists finite m, such that $t_n = \bar{t}$ for all $n > m$.*

Continuity of preference can now be expressed in the standard manner of requiring that there are no 'jumps in preference at infinity'.

***Continuity*:** For any pair of sequences of contracts $\langle c_n \rangle$ and $\langle c'_n \rangle$ that converge to $\bar{c}$ and $\bar{c}'$ respectively, if $c_n \succsim^i c'_n$ for all n, then $\bar{c} \succsim^i \bar{c}'$.

Finally, we require a minimum amount of non-degeneracy for the preferences with respect to the states in S. Formally, we require at least three states to be 'essential'.

Definition 2 *Fix $\succsim^i$. A state s in S is* essential *for $\succsim^i$ if there exists a pair of actions a and a' in A_0 and a contract c in C, such that '**if** $t(s)$ **then** a **else** c' $\succ^i$ '**if** $t(s)$ **then** a' **else** c'.*

We now have all the pieces for our representation result.

Theorem 1 (State-Dependent Expected Utility Representation) *Fix $\succsim^i$. If there are at least three essential states then the following are equivalent.*

1. *The relation $\succsim^i$ satisfies ordering, act-equivalence, sure-thing principle and continuity.*

2. *There exists for each state s in S a continuous utility function $u_s^i : A_0 \to \mathbb{R}$, such that the following additively-separable function represents $\succsim^i$:*

$$U^i(c) = \sum_{s \in S} u_s^i(f_c(s)). \tag{1}$$

   Moreover, the functions $u_s^i(\cdot)$ are unique up to multiplication by a common positive scalar $\alpha > 0$, and the addition of a state-dependent constant β_s.

Proof. Sufficiency of axioms. Consider the preference relation $\succsim_{\mathrm{F}}^i \subset A_0^{|S|} \times A_0^{|S|}$ over acts, induced by $\succsim^i$: $c \succsim^i c'$ implies $f_c \succsim_{\mathrm{F}} f_{c'}$. Consider a pair of acts, $f \succsim_{\mathrm{F}}^i f'$. By construction, there exists a pair of contracts c and c' such that $f_c = f$, $f_{c'} = f'$ and $c \succsim^i c'$. Now for any pair of contracts $\hat{c}$ and $\hat{c}'$, such that $f_{\hat{c}} = f$ and $f_{\hat{c}'} = f'$, it follows from act-equivalence that $\hat{c} \sim c$ and $\hat{c}' \sim c'$, and so by ordering we have $\hat{c} \succsim^i \hat{c}'$. Hence it is enough to obtain a representation $\hat{U}^i(f)$ of $\succsim_{\mathrm{F}}^i$, since we can then set $U^i(c) := \hat{U}^i(f_c)$.

It is straightforward to show that continuity of $\succsim^i$ implies that $\succsim_{\mathrm{F}}^i$ is continuous in the product topology of $A_0^{|S|}$; and that the sure-thing principle for $\succsim^i$ implies that $\succsim_{\mathrm{F}}^i$ satisfies the sure-thing principle for acts: that is, for any four acts f, f' f'' and f''', and any event $E \subset S$, if $f(s) = f''(s)$ and $f'(s) = f'''(s)$ for all $s \in E$, and $f(s) = f'(s)$ and $f''(s) = f'''(s)$ for all $s \notin E$ then $f \succsim_{\mathrm{F}}^i f'$ implies $f'' \succsim_{\mathrm{F}}^i f'''$. Hence by Debreu (1960, Thm 3) it follows that there exists an additive representation for $\succsim_{\mathrm{F}}^i$ as given in (1). Proof of necessity is straightforward and thus omitted. ■

Unless there is some exogenously given structure on the payoffs and their utility, in this formulation, as far as the 'ambiguity-free' preferences $\succsim^i$ represented by $U^i(.)$ are concerned, one cannot separate the probability of the state obtaining from the state-dependent utility (Karni, 1985). One cannot even determine the level of state-dependent utility. More precisely, it is the only the change in the state-dependent utility resulting from a change in the action taken in that state that is determined up to a positive scalar. From the statement of Theorem 1 it follows that if $u_s^i(\cdot)$ is a state-dependent utility that can be used for the representation in (1) then so can any function $\tilde{u}_s^i(a) = \alpha\tilde{u}_s^i(a) + \beta_s$, with $\alpha > 0$. But notice that for any pair of actions a and a' and any pair of states s and s', we have:

$$\frac{\tilde{u}_s^i(a) - \tilde{u}_s^i(a')}{\tilde{u}_{s'}^i(a) - \tilde{u}_{s'}^i(a')} = \frac{u_s^i(a) - u_s^i(a')}{u_{s'}^i(a) - u_{s'}^i(a')}.$$

We thus define the following equivalence class for state-dependent utilities.

Definition 3 *The state-dependent utility functions* $(u_s)_{s\in S}$ *and* $(\tilde{u}_s)_{s\in S}$ *are* cardinally equivalent *if there exists a positive scalar* $\alpha > 0$ *and vector of constants* $(\beta_s)_{s\in S}$*, s.t.* $\tilde{u}_s(a) \equiv \alpha u_s(a) + \beta_s$ *for all* s *in* S*.*

In the sequel we shall restrict attention to individuals whose preferences in the absence of ambiguity admit a state-dependent expected utility representation of the form given in (1). We shall identify such a preference relation by its state-dependent expected utility representation.

Definition 4 *Let* $\mathcal{U}$ *denote the set of* state-dependent expected utility functions *defined on the set of contracts* C *that take the form given in* (1)*.*

3.1 The Illustrative example continued

To illustrate the ideas and concepts we have introduced above, let us apply this framework to the example discussed in section 2.

Set $T_0 = \{t_1, t_2\}$, where t_1 corresponds to the test proposition, 'card drawn is red' and t_2 corresponds to test proposition, 'card drawn is black.' Formally, the state space S is given by $\{(1,1),\ (1,0),\ (0,1),\ (0,0)\}$ but for ease of exposition we denote it by $S = \{\boldsymbol{RB},\ \boldsymbol{R},\ \boldsymbol{B},\ \boldsymbol{W}\}$, derived from the table

	t_2	$\neg t_2$
t_1	$\boldsymbol{RB}$	$\boldsymbol{R}$
$\neg t_1$	$\boldsymbol{B}$	$\boldsymbol{W}$

.

The state $\boldsymbol{R}$ (respectively, $\boldsymbol{B}$, $\boldsymbol{W}$) corresponds to the state of the world in which the card drawn is 'red' (respectively, black, white), while the state $\boldsymbol{RB}$, is the 'impossible' state in which the card is both red and black.

The set of actions is the set of transfers from column to row, $A_0 = [-3, 3]$. The set of contracts can thus be characterized by state-contingent transfers $(A_0)^S$.

Without loss of generality, we take the endowment in the 'impossible' state $\boldsymbol{RB}$ to be $(0, 0)$. Hence the state-contingent endowments are given by

	State			
Ind.	$\boldsymbol{RB}$	$\boldsymbol{R}$	$\boldsymbol{B}$	$\boldsymbol{W}$
z_s^{Row}	0	3	1	2
z_s^{Col}	0	1	3	2

.

The state-dependent utility functions for the unambiguous preferences of Row and Col are given by

$$u_s^{Row}(a) = \begin{cases} 0 & \text{if } s = \boldsymbol{RB} \\ \frac{1}{3} v\left(a + z_s^{Row}\right) & \text{if } s \neq \boldsymbol{RB} \end{cases} \quad \text{and } u_s^{Col}(a) = \begin{cases} 0 & \text{if } s = \boldsymbol{RB} \\ \frac{1}{3} v\left(-a + z_s^{Col}\right) & \text{if } s \neq \boldsymbol{RB} \end{cases},$$

where $v(.)$ is the common continuous, strictly concave and strictly increasing utility function over final wealth for the two players and $(0, 1/3, 1/3, 1/3)$ can be interpreted as their common prior over the state space S. The state $\boldsymbol{RB}$ is not essential but the other three are all essential. Hence the preferences over contracts generated by the functionals

$$U^{Row}(c) = \frac{1}{3} v\left(f_c(\boldsymbol{R}) + 3\right) + \frac{1}{3} v\left(f_c(\boldsymbol{B}) + 1\right) + \frac{1}{3} v\left(f_c(\boldsymbol{W}) + 2\right), \tag{2}$$

$$U^{Col}(c) = \frac{1}{3} v\left(-f_c(\boldsymbol{R}) + 1\right) + \frac{1}{3} v\left(-f_c(\boldsymbol{B}) + 3\right) + \frac{1}{3} v\left(-f_c(\boldsymbol{W}) + 2\right), \tag{3}$$

satisfy the properties in Theorem 1.

From the symmetry of the setup it is immediate that the efficient (equal utility) risk-sharing contract is $\tilde{c}$ = '**if** $t_1 \wedge \neg t_2$ **then** -1 **else if** $t_2 \wedge \neg t_1$ **then** 1 **else** 0,' (that is, Row makes a transfer of 1 to Col if the card drawn is red, Col makes a transfer of 1 to Row if the card drawn is black, and otherwise no transfers are made.) The associated act is $f_{\tilde{c}} = (0, -1, 1, 0)$, and the subjective expected utility for both players is given by $U^{Row}(\tilde{c}) = U^{Col}(\tilde{c}) = v(2)$.[3]

4 Introducing ambiguity

Because we have chosen formally identical state spaces for the players, the test-interpretation of each player and the language of each player are identical. The distinction and the source of disputes is thus purely semantic. Disputes arise from the players disagreeing about the state that has obtained, or, equivalently, which tests have been satisfied. In this section we first introduce ambiguity by way of ambiguous tests and show how this makes some contracts 'ambiguous'. We then develop a model of ambiguity averse decision-makers.

4.1 Ambiguous tests and ambiguous contracts

We interpret a test t as *unambiguous* for the two players if, whenever one individual believes t is satisfied (respectively, not satisfied), she is confident that the other individual agrees with her. Hence for any contract of the form '**if** t **then** a **else** a',' both individuals are confident that they will agree the contract calls for action a when t is satisfied and a' when t is not satisfied.

Conversely, we interpret a test t as ambiguous for the two players, if whenever one player believes t is satisfied (respectively, not satisfied), she considers it is possible that the other player believes t is not satisfied (respectively, satisfied).[4] Whenever the two individuals disagree about whether or not the test t is satisfied, it means that for any simple contract of the form '**if** t **then** a **else** a',' one player views the contract as calling for action a while the other views the contract as calling for action a'.

[3] Since the state-dependent utility of both players is zero in $\boldsymbol{RB}$ regardless of the transfer made in that state, we shall normalize the transfer to be zero in that state.

[4] Notice that 'ambiguity' is a property of the test *with respect to the two players.* Although we do not explore the issue in this paper, it is possible and seems natural to us, that the set of ambiguous tests would change if either player were replaced by a different individual.

We presume that the set T of tests can be partitioned into the set of *unambiguous tests* T_U and the set of *ambiguous tests* $T_A = T - T_U$.

The set of unambiguous tests T_U is presumed to satisfy the following conditions:

(i) for all $t \in T$, if $\pi(t) = S$, then $t \in T_U$;

(ii) for all $t, t' \in T$, if $\pi(t) = \pi(t')$ and $t \in T_U$, then $t' \in T_U$;

(iii): for all t,$t' \in T_U$, $\neg t$ and $t \wedge t'$ are in T_U.

Condition (i) states that all certainty events are unambiguous. Condition (ii) states that if two tests are satisfied at precisely the same states, and one is unambiguous, then so is the other. The last condition (iii) states that the set of unambiguous tests is closed under conjunction and negation.

We can use the test interpretation to derive the set of unambiguous events.

Definition 5 *The set of* unambiguous events $\mathcal{E}_U \subseteq 2^S$ *is given by:*

$$\mathcal{E}_U = \{E \subseteq S : \exists t \in T_U \ s.t. \ \pi(t) = E\}.$$

The set of ambiguous events $\mathcal{E}^A = 2^S - \mathcal{E}_U$.

Lemma 2 *The set of unambiguous events $\mathcal{E}_U$ is an algebra of subsets of S that contains S and $\varnothing$. That is, it is closed under taking complements and intersection.*

Proof. Notice that for any $t \in T$, $t \wedge \neg t \in T_U$. Thus $\neg(t \wedge \neg t) \in T_U$. Since $\varnothing = \pi(t \wedge \neg t)$ and $S = \pi(\neg(t \wedge \neg t))$ it follows $\mathcal{E}_U$ contains S and $\varnothing$. Consider any pair of events E and E' in $\mathcal{E}_U$. By definition, there exists tests t and t' in T_U, such that $\pi(t) = E$ and $\pi(t') = E'$. But since T_U is closed under negation and conjunction, it follows the tests $\neg t$ and $t \wedge t'$ are also in T_U. And from the inductive construction of the interpretation test, it follows: $\pi(\neg t) = S - E$ and $\pi(t \wedge t') = E \cap E'$, as required. ■

For each $s \in S$, we can derive from the set of unambiguous tests the collection of possible states the other player may have determined as having obtained as follows.

Definition 6 (*Possibility Set*) *Suppose $T_U \subset T$, is the set of unambiguous tests. For each s*

in S, define the possibility-of-dispute set *associated with state s to be:*

$$P(s) := \{s' \in S : \textit{for each } t \in T_U, s \in \pi(t) \Rightarrow s' \in \pi(t)\}.$$

By construction, the set $P(s)$ comprises those states that cannot be distinguished from s by the outcome of an unambiguous test. Clearly, $s \in P(s)$ for each $s \in S$, so $P(s) \neq \varnothing$ for each $s \in S$. In addition $s' \in P(s)$ if and only if $P(s) = P(s')$, hence the collection of events $\{P(s)\}_{s \in S}$ forms a partition of S. We will refer to $\{P(s)\}_{s \in S}$ as the *possibility partition*.

For each $s \in S$ we can define the smallest unambiguous event $E(s)$ containing s by $E(s) := \bigcap_{E \in \{F \in \mathcal{E}_U : s \in F\}} E$. We have the following facts which show that $\{P(s)\}_{s \in S}$ is the finest unambiguous partition of S. More specifically, for each state s, the element of the possibility partition $P(s)$ is the smallest unambiguous event that contains s, and $P(s)$ is a singleton if and only if the test $t(s)$ associated with the state s is an unambiguous test.

Lemma 3 *For each $s \in S$: (a) $P(s) = E(s)$; (b) $P(s) = \{s\}$ if and only if $t(s) \in T_U$.*

Proof. (a) First we show $P(s) \subseteq E(s)$. Suppose that $s' \in P(s)$, but $s' \notin E(s)$. Observe that $E(s) \neq \varnothing$ follows from (i) in the definition of T_U. Hence, since $s' \notin E(s)$, there must be some $E \in \{F \in \mathcal{E}_U : s \in F\}$, and $s' \notin E$. Since $E \in \mathcal{E}_U$, there is a test $t \in T_U$ such that $\pi(t) = E$. Also, $s \in E$. Since $s' \in P(s)$, it follows from the definition of $P(s)$ that $s' \in \pi(t) = E$, which is a contradiction. Hence, we conclude that $P(s) \subseteq E(s)$.

Next we show that $E(s) \subseteq P(s)$. Suppose that $s' \in E(s)$, but $s' \notin P(s)$. Then there is some test $t \in T_U$ such that $s \in \pi(t)$ but $s' \notin \pi(t)$. Then $\pi(t)$ is an unambiguous event containing s but not containing s'. Hence $E(s) \subseteq \pi(t)$,and $s' \notin E(s)$, which is again a contradiction. Hence we conclude that $E(s) \subseteq P(s)$.

(b) (If) Clearly, $\{s\} \subseteq P(s)$ from the definition of $P(s)$. Next, since $t(s) \in T_U$ and $\pi(t(s)) = \{s\}$, it follows that if $s' \neq s$, then $s' \notin P(s)$, that is, $P(s) \subseteq \{s\}$.

(Only-if) Since $P(s) = \{s\}$, it follows that for each $s' \neq s$, there is a test $t' \in T_U$ such that $s \in \pi(t')$ and $s' \notin \pi(t')$. Since T_U is closed under conjunction by (iii), we can take the conjunction of these tests over $S - \{s\}$ to obtain an unambiguous test $t^* \in T_U$ that excludes everything but

s, that is, $\pi(t^*) = \{s\}$. Since $\pi(t(s)) = \{s\} = \pi(t^*)$, it follows from (ii) that $t(s)$ is unambiguous, that is, $t(s) \in \mathcal{E}_U$. ■

Notice that if a contract is measurable with respect to the possibility partition, although the individuals might disagree about the actual state that has obtained, they will never disagree about which action the contract prescribes. Hence such contracts are viewed as *unambiguous.*

Definition 7 *A contract is* unambiguous *if for all for all $s, s' \in S$, $P(s) = P(s') \Rightarrow f_c(s) = f_c(s')$. We denote by C_U the set of unambiguous contracts.*

4.2 The effect of ambiguity on an individual's preferences over contracts.

To model the effect that the presence of ambiguity has on preferences over contracts, consider an individual i who in the absence of any ambiguity has preferences over contracts that admit a representation $U^i \in \mathcal{U}$. If T_A is the set of tests she views as ambiguous, then as we have shown above, the possibility partition $\{P(s)\}_{s \in S}$ corresponds to the coarsest unambiguous partition of S. In particular, when individual i believes that the state is s, she considers it possible that the other party may believe any element of $P(s)$ obtains. Hence in terms of a given contract c, this possibility of dispute generates *ambiguity* about the action that will actually be implemented. Depending upon which interpretation is followed, the action might conceivably be any member of the set $\{f_c(s') : s' \in P(s)\}$.

Here we are using 'ambiguity' in the ordinary language sense of the term; the meaning of the contract is unclear because different interpretations may arise. This ambiguity in language gives rise to ambiguous preferences in the sense commonly used in the decision theory literature. A dispute is outside the framework of the formal language in which the contracts are written, and thus outside the associated state space and additive state-dependent utility structure. Hence, reasoning about disputes cannot be undertaken in terms of numerical probabilities derived from individual preferences over unambiguous contracts. Rather we shall model the individual's state-dependent preferences in the presence of possible disputes as ambiguity averse preferences.

Although individual i who believes the state is s does not have a fully developed probability distribution for the results of a dispute, she can do no worse than accept the least favorable action implied by the contract in the set of possible interpretations of the tests by j at s, that is, in the set $\{f_c(s') : s' \in P(s)\}$. That is, for each state s the contract c induces a lower bound of utility for individual i of $\min_{s' \in P(s)} u_s^i(f_c(s'))$. This provides a state-dependent bound to any loss from disputes. Hence, one possible way to model the potential loss from dispute is to assign a decision weight to this worst-case outcome. This reasoning corresponds to one of the most commonly applied models of ambiguity averse preferences, the ε-contamination model.[5]

This model is a special case of the multiple prior expected utility model, in which the set of multiple priors corresponds to a particular type of neighborhood of 'radius' $\varepsilon^i \in [0,1]$ 'centered' around a given prior. To translate this to our setting, notice that when individual i assesses the state as being s, her posterior belief is that according to her interpretation of the tests, the action $f_c(s)$ should be undertaken. However, i also perceives the possibility that j may have judged that any of the states in $P(s)$ may have obtained. Hence, as we have already noted, the ambiguity she faces is that the action undertaken could be any member of the set $\{f_c(s') : s' \in P(s)\}$. If we let $\Delta(\{f_c(s') : s' \in P(s)\})$ denote the set of probability measures over this set of actions, and if we let ε^i be the decision-weight she assigns to the ambiguity she faces, then her ε-contaminated subjective expected utility in state s is given by

$$\begin{aligned} & \left(1-\varepsilon^i\right) u_s^i(f_c(s)) + \varepsilon^i \min_{p \in \Delta(\{f_c(s') : s' \in P(s)\})} \sum_{a \in \{f_c(s') : s' \in P(s)\}} p(a)\, u_s^i(a) \\ = \; & \left(1-\varepsilon^i\right) u_s^i(f_c(s)) + \varepsilon^i \min_{s' \in P(s)} u_s^i(f_c(s')). \end{aligned} \tag{4}$$

Adjusting the representation in (1) to allow for consideration of the possibilities of dispute in terms of an ε-contamination adjustment of the state-dependent expected utility, we have that individuals, when taking into account the possibility of dispute, rank contracts according to the following function:

[5] The approach here may can viewed as a state-dependent extension of Kopylov (2008).

$$
\begin{aligned}
V^i\left(c;T_A,\varepsilon^i\right) &= \sum_{s\in S}\left[\left(1-\varepsilon^i\right)u_s^i\left(f_c\left(s\right)\right)+\varepsilon^i\min_{s'\in P(s)}u_s^i\left(f_c\left(s'\right)\right)\right], \\
&= \left(1-\varepsilon^i\right)U^i\left(c\right)+\varepsilon^i\sum_{s\in S}\min_{s'\in P(s)}u_s^i\left(f_c\left(s'\right)\right). \qquad (5)
\end{aligned}
$$

We refer to a preference relation over C that is generated by the function $V^i\left(\cdot;T_A,\varepsilon^i\right)$ as an ε*-contaminated state-dependent utility maximizer*. Obviously, $V^i\left(\cdot;T_A,\varepsilon^i\right)=U^i\left(\cdot\right)$ whenever $T_A=\emptyset$ or $\varepsilon^i=0$. Furthermore, for any unambiguous contract c in C_U, $V^i\left(c;T_A,\varepsilon^i\right)=U^i\left(c\right)$

Definition 8 *An* ε-contaminated state-dependent utility maximizer *is a preference relation that admits a representation of the form in* (5). *A member of this set is characterized by the triple* $\left(T_A,U^i,\varepsilon^i\right)\in T\times\mathcal{U}\times[0,1]$.

4.3 The illustrative example continued

Returning to our illustrative example, recall $T_0=\{t_1,t_2\}$, where t_1 is the test proposition 'card drawn is red' and t_2 is test proposition 'card drawn is black'. As Row only sees the top of a card and Col only sees the bottom, it follows from the definitions above that T_U is the closure under $\neg$ and $\wedge$ of $\{t_1\wedge t_2, t_1\wedge\neg t_2,\neg t_1\}$. Hence the possibility of disputes sets associated with each of the four states are given by $P\left(\boldsymbol{RB}\right)=\{\boldsymbol{RB}\}$, $P\left(\boldsymbol{R}\right)=\{\boldsymbol{R}\}$, $P\left(\boldsymbol{B}\right)=P\left(\boldsymbol{W}\right)=\{\boldsymbol{B},\ \boldsymbol{W}\}$.

If we suppose that the ambiguity aversion parameter $\varepsilon^i=\varepsilon>0$ is the same for both players, then the ε-contamined state-dependent utility preferences over contracts for Row and Col are generated by the functionals:

$$
\begin{aligned}
V^{Row}\left(c;T_A,\varepsilon\right) &= (1-\varepsilon)\,U^{Row}\left(c\right) \\
&\quad +\frac{\varepsilon}{3}\left[v\left(f_c\left(\boldsymbol{R}\right)+3\right)+\min_{s\in\{B,W\}}v\left(f_c\left(s\right)+1\right)+\min_{s\in\{B,W\}}v\left(f_c\left(s\right)+2\right)\right], \qquad (6) \\
V^{Col}\left(c;T_A,\varepsilon\right) &= (1-\varepsilon)\,U^{Col}\left(c\right) \\
&\quad +\frac{\varepsilon}{3}\left[v\left(-f_c\left(\boldsymbol{R}\right)+1\right)+\min_{s\in\{B,W\}}v\left(-f_c\left(s\right)+3\right)+\min_{s\in\{B,W\}}v\left(-f_c\left(s\right)+2\right)\right] (7)
\end{aligned}
$$

Some aspects of the solution are noteworthy. The players' (*ex ante*) preference for signing a given hedging contract will be stronger the more risk-averse they are, that is, the stronger their preference for the non-stochastic allocation over the original endowment. Their preference for signing a hedging contract will be less the more weight they place on the possibility of different interpretations giving rise to disputes. Thus risk and ambiguity work in opposite directions. This result applies generally to problems involving ambiguous risk sharing contracts.

5 The bargaining problem

We follow Rubinstein, Safra and Thomson's (1992) recasting of the classic two-agent bargaining problem in terms of the bargainers' risk preferences. We take the set of alternatives over which the (*ex ante*) bargaining is conducted to be the set of contracts C, characterized in section 3. We further assume that there is a designated contract $c_0 \in C$, which we take to be the default or disagreement action that will result should the bargaining process break down and no agreement is reached.

As C and c_0 will be fixed throughout, we shall, following Rubinstein et al. (1992), identify the bargaining problem with the pair of preferences relations of the bargainers over $\Delta_0(C)$, the set of simple probability distributions or lotteries over C. Formally, $\Delta_0(C)$ is the set of functions $L : C \to [0,1]$, satisfying $\sum_{c\in C} L(c) = 1$ and where $L(c)$ is interpreted as the probability that the contract c will obtain under L.

So that the problem is not vacuous, we assume there exists a contract $\hat{c}$ in C such that $V^i\left(\hat{c}; T_A, \varepsilon^i\right) > V^i\left(c_0; T_A, \varepsilon^i\right)$, $i = 1, 2$. That is, $\hat{c}$ (strictly) Pareto dominates c_0.

Part of the motivation of Rubinstein et al. for recasting the bargaining problem in terms of the bargainers' risk preferences was to allow for risk preferences that did not necessarily conform to expected utility theory. For simplicity, however, we shall restrict attention to bargainers whose (risk) preferences over $\Delta_0(C)$ conform to expected utility theory. We further assume that the representation $V^i\left(\cdot; T_A, \varepsilon^i\right)$ is also the von Neumann–Morgenstern utility index for the expected utility representation of the entire preference relation.[6] That is, bargainer i strictly prefers the

[6] We view this as being consistent with the approach we have taken throughout. Preferences over objective or 'unambiguous' subjective randomizations conform to expected utility theory. Departures from the standard model stem only from non-probabilistic uncertainty such as the ambiguity the individual faces concerning the possibility of disputes arising from different interpretations of tests.

lottery L over contracts to the lottery L' over contracts, if and only if:

$$\sum\nolimits_{c\in C} L(c)\, V^i\left(c; T_A, \varepsilon^i\right) > \sum\nolimits_{c\in C} L(c)\, V^i\left(c; T_A, \varepsilon^i\right). \tag{8}$$

Thus a bargaining problem in our set-up can be identified by a quintuple $\left(T_A, U^1, \varepsilon^1, U^2, \varepsilon^2\right)$. Set $\mathcal{B} := T \times (\mathcal{U} \times [0,1])^2$ to denote the class of Bargaining problems for the analysis.

The following is Grant and Kajii's (1995) restatement of Rubinstein et al (1992)'s definition of the *Ordinal Nash Outcome* for a bargaining problem.

Definition 9 *Fix a bargaining problem* $\left(T_A, U^1, \varepsilon^1, U^2, \varepsilon^2\right)$ *in* $\mathcal{B}$. *We say that* bargainer i can appeal against the proposal c, *if there is an alternative contract* c' *and a probability* p *in* $[0,1]$ *such that* $pV^i\left(c'; T_A, \varepsilon^i\right) + (1-p)\, V^i\left(c_0; T_A, \varepsilon^i\right) > V^i\left(c'; T_A, \varepsilon^i\right)$ *and* $V^j\left(c'; T_A, \varepsilon^j\right) > pV^j\left(c; T_A, \varepsilon^j\right) + (1-p)\, V^j\left(c_0; T_A, \varepsilon^i\right)$. *An* ordinal Nash outcome *is a contract* c^* *in* C *against which neither bargainer can appeal.*

The interpretation given by Rubinstein et al to a proposal c being vulnerable to appeal by bargainer i with the alternative c', is as follows: The first inequality implies that i would prefer to go with c' rather than accept the proposal c even if by going with c' meant that, with probability $(1-p)$, negotiations might breakdown, resulting in the default contract c_0 being undertaken. The second inequality implies that j would prefer to concede c' rather than stick with c if, by sticking with c, she risked, with probability $(1-p)$, negotations breaking down, again resulting in the default contract c_0 being undertaken. If such a configuration of preferences hold in a bargaining situation, Rubinstein et al argue that j would not stay with c but would concede c' to i. Hence the solution to the bargainer problem should be an outcome that is not vulnerable to any such appeal by either of the bargainers.

Rubinstein et al show that, for the case of expected utility maximizers, the ordinal Nash solution corresponds to the outcome that maximizes the product of utility gains over the default action. Although we agree with Rubinstein et al. that the product of utility gains is difficult to interpret, we also agree with Grant and Kajii's (1995) assessment that the program of finding the contract that maximizes the product of utility gains is a useful operational method for determining the ordinal Nash outcome as well as highlighting the way in which it is a natural preference based analog of the original 'utility' based definition.[7]

To develop this analogy to the 'utility' based definition, it is useful to define the *cardinal bargaining problem* induced by those preferences in the following way.

Definition 10 *Fix a bargaining problem $\left(T_A, U^1, \varepsilon^1, U^2, \varepsilon^2\right)$ in $\mathcal{B}$. The* cardinal bargaining problem *associated with $\left(T_A, U^1, \varepsilon^1, U^2, \varepsilon^2\right)$ is the set $B \subset \mathbb{R}_2$, given by*

$$B = \left\{(v_1, v_2) : \exists c \in C, (v_1, v_2) \leq \left(V^1(c), V^2(c)\right)\right\}.$$

Notice that B is comprehensive by construction. Since A_0 is compact, it follows that C is compact as well, and hence it follows by the construction of B that it is closed. In order for the bargaining problem to be well posed we assume the bargaining problem exhibits the following property.

Definition 11 (C-Convexity) *A bargaining problem $\left(T_A, U^1, \varepsilon^1, U^2, \varepsilon^2\right)$ in $\mathcal{B}$ exhibits* C-convexity *if for any pair of contracts c and c' in C, there exists a contract c'' in C such that*

$$V^i\left(c''; T_A, \varepsilon^i\right) \geq \frac{1}{2} V^i\left(c; T_A, \varepsilon^i\right) + \frac{1}{2} V^i\left(c'; T_A, \varepsilon^i\right),\ i = 1, 2.$$

A sufficient condition for a bargaining problem to exhibit C-convexity, is for the state-dependent utility functions $u_s^i(.)$ to be concave in a.[8]

As the name suggests, a bargaining problem that exhibits C-convexity has associated with it a *convex* cardinal bargaining problem.

Lemma 4 *If $\left(T_A, U^1, \varepsilon^1, U^2, \varepsilon^2\right)$ in $\mathcal{B}$ is a C-convex bargaining problem then the associated cardinal bargaining problem B is convex.*

---

[7] Notice that although preferences are over lotteries defined on the set of contracts, the ordinal Nash outcome is required to be a contract (more precisely, a degenerate lottery over contracts). So the question remains, in applying the approach of Rubinstein et al. to bargaining in our contractual setting, whether an ordinal Nash outcome exists and, if so, whether or not it is (essentially) unique.

[8] This holds naturally for risk-sharing contracts in which the action $a \in A_0 \subset \mathbb{R}$ corresponds to a transfer of size a from bargainer 2 to bargainer 1, and $u_s^i(a) = \pi_s v\left(a \times (-1)^{i-1} + z_s^i\right)$ is the probability weighted utility of bargainer i's final wealth in state s after the transfer has been made. Concavity of $u_s^i(.)$ then follows naturally from risk aversion (that is, concavity of v the utility index over wealth).

Proof. Fix an arbitrary pair (v_1, v_2) and (v_1', v_2') in B. To establishthat B is convex it is sufficient to show that $\left(\frac{1}{2}v_1 + \frac{1}{2}v_1', \frac{1}{2}v_2 + \frac{1}{2}v_2'\right)$ is also in B. Since (v_1, v_2) and (v_1', v_2') are both in B, it follows from the definition of B that there exists contracts c and c' in C, such that $\left(V^1(c), V^2(c)\right) \geq (v_1, v_2)$ and $\left(V^1(c'), V^2(c')\right) \geq (v_1', v_2')$. By C-convexity there exists a contract c'' in B, such that $V^i\left(c''; T_A, \varepsilon^i\right) \geq \frac{1}{2}V^i\left(c; T_A, \varepsilon^i\right) + \frac{1}{2}V^i\left(c'; T_A, \varepsilon^i\right)$, $i = 1, 2$. Hence,

$$\begin{aligned}\left(V^1(c''), V^2(c'')\right) &\geq \left(\frac{1}{2}V^1(c) + \frac{1}{2}V^1(c'), \frac{1}{2}V^2(c) + \frac{1}{2}V^2(c')\right) \\ &\geq \left(\frac{1}{2}v_1 + \frac{1}{2}v_1', \frac{1}{2}v_2 + \frac{1}{2}v_2'\right),\end{aligned}$$

as required. ■

Putting this all together, we have that a C-convex bargaining problem in $\mathcal{B}$ is a well posed problem.

Proposition 5 *Suppose a bargaining problem* $\left(T_A, U^1, \varepsilon^1, U^2, \varepsilon^2\right)$ *in* $\mathcal{B}$ *is C-convex. Then there exists an ordinal Nash outcome* c^* *which is the solution to the program,*

$$\max_{c \in C}\left[V^1\left(c; T_A, \varepsilon^i\right) - V^1\left(c_0; T_A, \varepsilon^1\right)\right]\left[V^2\left(c; T_A, \varepsilon^2\right) - V^2\left(c_0; T_A, \varepsilon^2\right)\right]. \tag{9}$$

Moreover, c^* *is unique in terms of the equivalence classes of* $\left(T_A, U^1, \varepsilon^1\right)$ *and* $\left(T_A, U^2, \varepsilon^2\right)$*. That is, for any contract* $\hat{c}$ *that is a solution of* (9), $V^1\left(\hat{c}; T_A, \varepsilon^1\right) = V^1\left(c^*; T_A, \varepsilon^1\right)$ *and* $V^2\left(\hat{c}; T_A, \varepsilon^2\right) = V^2\left(c^*; T_A, \varepsilon^2\right)$.

5.1 Relating the ordinal Nash solution to the alternating offers game.

Rubinstein et al. (1992) also point out the connection between the strategic alternating offers model of Rubinstein (1982) and the ordinal Nash solution. Recall the version of the infinite alternating offers model in which at the end of each period there is a probability $1 - p > 0$ of breakdown if agreement has not already been reached. Rubinstein et al. note that for the case of expected utility bargainers there is a unique subgame perfect equilibrium. In our setting the unique[9] subgame perfect equilibrium is characterized by two contracts, $c^1(p)$ and $c^2(p)$ satisfying:

$$\begin{aligned} pV^1\left(c^1(p); T_A, \varepsilon^1\right) + (1-p)V^1\left(c_0; T_A, \varepsilon^1\right) &= V^1\left(c^2; T_A, \varepsilon^1\right) \\ \text{and } pV^2\left(c^2(p); T_A, \varepsilon^2\right) + (1-p)V^2\left(c_0; T_A, \varepsilon^2\right) &= V^2\left(c^1; T_A, \varepsilon^2\right). \end{aligned}$$

[9] Strictly speaking, we mean unique in terms of the equivalence classes of the bargainers' preferences.

Bargainer 1 always offers $c^1(p)$ and accepts any offer by bargainer 2 of a contract c satisfying $V^1(c) \geq V^1\left(c^2(p)\right)$, while bargainer 2 always offers $c^2(p)$ and accepts any offer by bargainer 1 of a contract satisfying $V^2(c) \geq V^2\left(c^1(p)\right)$.

As an immediate corollary of Rubinstein et al's (1992, Proposition 4, p1183), in the limit as $p \to 1$, both $c^1(p) \to c^*$ and $c^2(p) \to c^*$.[10] That is, in the limit as the risk of breakdown at the end of each period goes to zero, the unique subgame perfect equilibrium outcome converges to the ordinal Nash outcome.

5.2 The illustrative example continued

Taking $c_0 = 0$ (that is, no transfer is made), and denoting $V^{Row}(c_0; T_A, \varepsilon) = V^{Col}(c_0; T_A, \varepsilon) = \bar{u}$, the bargaining problem for our illustrative example developed in section 4.3 corresponds to finding a contract c that maximizes

$$\max_{\left\langle (f_c(\boldsymbol{R}), f_c(\boldsymbol{B}), f_c(\boldsymbol{W})) \in [-3,3]^3 \right\rangle} \left(V^{Row}(c; T_A, \varepsilon) - \bar{u}\right)\left(V^{Col}(c; T_A, \varepsilon) - \bar{u}\right).$$

The solution to this problem c^* with associated state-contingent transfers $(0, f_{c^*}(\boldsymbol{R}), f_{c^*}(\boldsymbol{B}), f_{c^*}(\boldsymbol{W}))$ (and assuming an interior solution with $f_{c^*}(\boldsymbol{R}) \leq f_{c^*}(\boldsymbol{W}) \leq f_{c^*}(\boldsymbol{B})$), satisfies the first-order conditions:

$$\begin{aligned}
f_{c^*}(\boldsymbol{R}) &: \frac{\frac{1}{3} v'(f_{c^*}(\boldsymbol{R}) + 3)}{(V^{Row}(c^*; T_A, \varepsilon) - \bar{u})} - \frac{\frac{1}{3} v'(-f_{c^*}(\boldsymbol{R}) + 1)}{(V^{Col}(c^*; T_A, \varepsilon) - \bar{u})} = 0 \\
f_{c^*}(\boldsymbol{B}) &: \frac{\frac{1}{3}(1-\varepsilon) v'(f_{c^*}(\boldsymbol{B}) + 1)}{(V^{Row}(c^*; T_A, \varepsilon) - \bar{u})} - \frac{\frac{1}{3}\left[v'(-f_{c^*}(\boldsymbol{B}) + 3) + \varepsilon v'(-f_c(\boldsymbol{B}) + 2)\right]}{(V^{Col}(c^*; T_A, \varepsilon) - \bar{u})} = 0 \\
f_{c^*}(\boldsymbol{W}) &: \frac{\frac{1}{3}\left[v'(f_{c^*}(\boldsymbol{W}) + 2) + \varepsilon v'(f_{c^*}(\boldsymbol{W}) + 1)\right]}{(V^{Row}(c^*; T_A, \varepsilon) - \bar{u})} - \frac{\frac{1}{3}(1-\varepsilon) v'(-f_{c^*}(\boldsymbol{W}) + 2)}{(V^{Col}(c^*; T_A, \varepsilon) - \bar{u})} = 0.
\end{aligned}$$

Or rearranging, we obtain:

$$\begin{aligned}
& \frac{v'(f_{c^*}(\boldsymbol{R}) + 3)}{v'(-f_{c^*}(\boldsymbol{R}) + 1)} = \frac{(1-\varepsilon) v'(f_{c^*}(\boldsymbol{B}) + 1)}{v'(-f_{c^*}(\boldsymbol{B}) + 3) + \varepsilon v'(-f_c(\boldsymbol{B}) + 2)} \\
= \; & \frac{v'(f_{c^*}(\boldsymbol{W}) + 2) + \varepsilon v'(f_{c^*}(\boldsymbol{W}) + 1)}{(1-\varepsilon) v'(-f_{c^*}(\boldsymbol{W}) + 2)} = \frac{V^{Row}(c^*; T_A, \varepsilon) - \bar{u}}{V^{Col}(c^*; T_A, \varepsilon) - \bar{u}}.
\end{aligned} \tag{10}$$

It is convenient to consider an unbounded extension of the problem in which transfers can be arbitrarily large in absolute value. The associated cardinal problem $\hat{B}$ for this extension is

[10] Again strictly speaking, the two offers can converge to any contract $\hat{c}$ for which $\hat{c} \sim^1 c^*$ and $\hat{c} \sim^2 c^*$.

symmetric. That is, if $(v_1, v_2) \in \hat{B}$ then $(v_2, v_1) \in \hat{B}$. To see this, notice that for any contract c with associated act f_c which generates the point $\left(V^{Row}(c; T_A, \varepsilon), V^{Col}(c; T_A, \varepsilon)\right) = (v_1, v_2)$ in $\hat{B}$, we can consider the 'complementary' contract $\bar{c}$ with an associated act $f_{\bar{c}}$, given by $f_{\bar{c}}(s) = -f_c(s) + z_s^{Col} - z_s^{Row}$. This generates the point (v_2, v_1), as is required for $\hat{B}$ to be symmetric.

Now symmetry of $\hat{B}$ implies that $V^{Row}(c^*) - \bar{u} = V^{Col}(c^*) - \bar{u}$.

We see immediately from (10) that for $\varepsilon = 0$, the solution is $(0, -1, 1, 0)$. That is, when no weight is given to the possibility of dispute, the solution is the full risk-sharing contract described in section 2.

To see what happens for $\varepsilon > 0$, we have from (10) that $f_{c^*}(\boldsymbol{R}) = 1$ and furthermore whenever $f_{c^*}(\boldsymbol{B}) > 1/2 > f_{c^*}(\boldsymbol{W})$ holds, $f_{c^*}(\boldsymbol{B})$ and $f_{c^*}(\boldsymbol{W})$ are the unique solutions to:

$$\frac{1}{(1-\varepsilon)} \frac{v'(-f_{c^*}(\boldsymbol{B}) + 3)}{v'(f_{c^*}(\boldsymbol{B}) + 1)} + \frac{\varepsilon}{(1-\varepsilon)} \frac{v'(-f_{c^*}(\boldsymbol{B}) + 2)}{v'(f_{c^*}(\boldsymbol{B}) + 1)} = 1 \tag{11}$$

$$\frac{1}{(1-\varepsilon)} \frac{v'(f_{c^*}(\boldsymbol{W}) + 2)}{v'(-f_{c^*}(\boldsymbol{W}) + 2)} + \frac{\varepsilon}{(1-\varepsilon)} \frac{v'(f_{c^*}(\boldsymbol{W}) + 1)}{v'(-f_{c^*}(\boldsymbol{W}) + 2)} = 1. \tag{12}$$

respectively. Notice that the LHS of (11) is increasing in $f_{c^*}(\boldsymbol{B})$ and the LHS of (12) is decreasing in $f_{c^*}(\boldsymbol{W})$, so while $f_{c^*}(\boldsymbol{B}) > \frac{1}{2} > f_{c^*}(\boldsymbol{W})$ the solution is well-defined for each corresponding ε.

For the critical value $\hat{\varepsilon}$ for which $f_{c^*}(\boldsymbol{B}) = f_{c^*}(\boldsymbol{W}) = \frac{1}{2}$, that is, $\hat{\varepsilon} = \left[1 - \left(v'\left(\frac{5}{2}\right) / v'\left(\frac{3}{2}\right)\right)\right] / 2$, the optimal contract is the *unambiguous contract:* '**if** t_1 **then** -1 **else** $\frac{1}{2}$,' , that is, if the card drawn is red then Row pays Col 1, and otherwise Col pays Row $\frac{1}{2}$. This remains the optimal contract for any $\varepsilon > \hat{\varepsilon}$, since the state-contingent act associated with this contract $\left(0, -1, \frac{1}{2}, \frac{1}{2}\right)$ satisfies

$$\frac{v'(f_{c^*}(\boldsymbol{R}) + 3)}{v'(-f_{c^*}(\boldsymbol{R}) + 1)} = \frac{v'(f_{c^*}(\boldsymbol{B}) + 1) + v'(f_{c^*}(\boldsymbol{W}) + 2)}{v'(-f_{c^*}(\boldsymbol{B}) + 3) + v'(-f_{c^*}(\boldsymbol{W}) + 2)} = \frac{V^{Row}(c^*; T_A, \varepsilon) - \bar{u}}{V^{Col}(c^*; T_A, \varepsilon) - \bar{u}},$$

the first-order conditions for the optimal *unambigous* contract.

6 Coordination in the presence of ambiguity

Ambiguity can arise in many forms and settings. In this section, we consider a symmetric situation in which two agents face a coordination problem with ambiguity in the sense that the appropriate actions to coordinate upon are ambiguous. Without ambiguity, it would be a pure coordination game, and the players could easily coordinate appropriately in each state. Ambiguity turns the coordination game into a 'battle of the sexes' game whenever the agents disagree about the result of a test.

The benefit of a contract in this setting turns out to be two-fold. As usual, contracting ensures coordination on the appropriate actions of the players. Additionally, however, it reduces the downside of ambiguity by constraining the players to implement something within the contract. In this way, the agents can use contracting to avoid costly and inefficient disputes that might arise under purely non-cooperative behavior.

In terms of the effect of ambiguity on the benefits from contracting, we observe a non-monotonic relationship. Starting with no ambiguity, the parties do not value contracting, since they will agree on the contingency (state of the world) and on the appropriate joint coordinated actions. These will be apparent and self-enforcing in each contingency. When a slight amount of ambiguity appears, however, the contract suddenly has value because it reduces the downside outcome by constraining the agents to implement the contract. As the level of ambiguity increases, the benefits from coordination are diminished and, for sufficiently high levels of ambiguity, the players resort to the safe option which is self-enforcing and thus the benefits of contracting diminish to zero again.

To make this more precise, suppose each party $i = 1, 2$ has an action set $A_i = \{\alpha, \beta, \gamma\}$ and that there is a unique primitive test proposition t. Thus, $T_0 = \{t\}$, and the state space $S = \{(0), (1)\}$, where state $s = (1)$ corresponds to the state in which t is true. The payoff for party i contingent on their interpretations of the result of test t are given by the following table:

Payoff to i

t is true

i's action \ j's action	α	β	γ
α	2	-1	-1
β	-1	0	-1
γ	1	1	1

$\neg t$ is true

i's action \ j's action	α	β	γ
α	0	-1	-1
β	-1	2	-1
γ	1	1	1

The 'safe' action for each player is γ as it guarantees the agent a payoff of 1 regardless of his opponent's choice and the state. In the absence of contracting, the players are presumed to choose their state contingent actions independently in a game associated with these payoffs (a game where nature moves first and determines with equal probability whether t or $\neg t$ holds, followed by simultaneous choices by the two players of an action from $\{\alpha, \beta, \gamma\}$). In this non-cooperative case, a strategy for party $i = 1, 2$ can be expressed as a contingent action choice of the form 'if t then a_i else a_i'' for some $a_i, a_i' \in A_i$. Let $\Sigma_1 = \Sigma_2 = \Sigma$ denote the set of independent strategies for the two parties.

If the situation is one with no ambiguity, then any symmetric strategy profile of the form 'if t then a else b' where $a \in \{\alpha, \gamma\}$ and $b \in \{\beta, \gamma\}$ is a subgame perfect equilibrium of the game. The Pareto efficient equilibrium is (σ^*, σ^*) where $\sigma^* =$ **'if** t **then** α **else** β', yielding an equilibrium payoff of 2 to each party.

One might argue that the benefit for the parties in committing themselves to a contract c^* is that they remove all strategic uncertainty. But even without a contract, a number of game theorists have argued (most notably, Harsanyi and Selten, 1988) that in such a situation where the strategic interests of the parties are so aligned, communication between the parties alone should be sufficient to ensure that an efficient equilibrium would be played out. When there is no ambiguity, we are in agreement that the efficient outcome would obtain without contracting.

However, ambiguity changes matters. The safe strategy profile (σ_0, σ_0), where $\sigma_0 =$ '**if** t **then** γ **else** γ' still guarantees the players a payoff of 1. By contrast, under ambiguity, the strategy profile (σ^*, σ^*) may lead the parties to fail to coordinate on the same action, producing a payoff of -1.

The set of contracts we will consider here consists of the set of contingent action profiles of the form '**if** t **then** (a_1, a_2) **else** (a_1', a_2')' for some (a_1, a_2) & (a_1', a_2') in $A_1 \times A_2$. Take the disagreement contract c_0 to be the 'safe' contract $c_0 =$ '**if** t **then** (γ, γ) **else** (γ, γ),' which like the safe strategy profile (σ_0, σ_0) guarantees both players a payoff of 1. In addition, in the absence of ambiguity, the Pareto efficient symmetric contract $c^* =$ '**if** t **then** (α, α) **else** (β, β)' generates the payoff of 2 to each party.

We presume that t is an ambiguous test, so $P(s) = S$ for each $s \in S$, and $\varepsilon_1 = \varepsilon_2 = \varepsilon > 0$. Hence, party i's evaluation of the non-cooperative strategy profile

$$(\sigma_i, \sigma_j) = \left(\text{‘}\mathbf{if}\ t\ \mathbf{then}\ a_i\ \mathbf{else}\ a_i'\text{’}, \text{‘}\mathbf{if}\ t\ \mathbf{then}\ a_j\ \mathbf{else}\ a_j'\text{’}\right),$$

is given by

$$V(\sigma_i, \sigma_j) = (1-\varepsilon)\left[u_i(a_i, a_j, 1) + u_i\left(a_i', a_j', 0\right)\right] + \varepsilon\left[\min_{\hat{a} \in \left\{a_j, a_j'\right\}} u_i(a_i, \hat{a}, 1) + \min_{\hat{a} \in \left\{a_j, a_j'\right\}} u_i(a_i', \hat{a}, 0)\right].$$

Thus, the $\varepsilon-$contaminated state-dependent expected utility of the strategy profile (σ^*, σ^*) is $V(\sigma^*, \sigma^*) = (1-\varepsilon)\,2 + \varepsilon(-1)$. Notice that the lower bound on playing α (respectively, β) when t (respectively, $\neg t$) is true, occurs when the other party perceives $\neg t$ (respectively, t) is true and so plays β (respectively, α). Action choices are mismatched leading to an outcome of -1.

However, the $\varepsilon-$contaminated expected utility of the contract c^* is given by $V(c^*) = (1-\varepsilon)\,2 + \varepsilon \times 0$ since these contracts ensure coordination of the actions and limit the worst case to 0 instead of -1. Here we see a benefit from contracting over non-cooperative play in the face of ambiguity. The contract, by constraining the action choices of the two parties to be coordinated, avoids the negative payoff associated with mismatching and only leaves open the less unpalatable outcome of coordinating on the ‘wrong’ action.

As the introduction to this section foreshadowed, we have

$$\begin{aligned} V(c^*) &> V(\sigma^*, \sigma^*) \text{ for all } \varepsilon > 0, \\ \text{and } V(c^*) &> V(\sigma_0, \sigma_0) \text{ if and only if } \varepsilon < 1/2. \end{aligned}$$

Thus we see that when the ambiguity parameter ε is positive but not too large, the agents will engage in contracting to coordinate their actions and avoid mismatching. However, when $\varepsilon > 1/2$ the agents resort to the safe contract c_0, which can be implemented without an explicit contract by the safe strategy profile (σ_0, σ_0).

7 Concluding comments

We have provided a formal model for incorporating ambiguity into decision making. The ambiguity in our model arises from the bounded rationality of the players which is expressed as limited abilities to perform tests over the possible contingencies. This limitation results in each player having a limited individual description of the world.

Contracting was restricted in this context to the types of test based contingent plans described in Blume et al. (2006). In this context we were able to show how ambiguity can affect incentives for risk sharing, and the desirability of contracts.

The representation of ambiguity proposed here suggests new approaches to a range of issues in contract theory. Some of these issues have proved difficult to address using approaches based on unbounded rationality, or on arbitrary constraints on rationality. In the case of risk sharing, we have shown that ambiguity may lead players to prefer incomplete risk sharing to possibly ambiguous contracts. On the other hand, in coordination games, contracting may represent a partial solution to ambiguity, requiring matching actions even when players disagree about the interpretation of the relevant test.

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