Spreadsheets in Education (eJSiE)

Volume 4 | Issue 1 Article 1

3-31-2010

Spreadsheet Implementations for Solving Boundary-Value Problems in Electromagnetics

Mark A. Lau
Universidad del Turabo, mlau@suagm.edu

Sastry P. Kuruganty
Universidad del Turabo, ut_skurugant@suagm.edu

Follow this and additional works at: http://epublications.bond.edu.au/ejsie

Recommended Citation
Lau, Mark A. and Kuruganty, Sastry P. (2010) "Spreadsheet Implementations for Solving Boundary-Value Problems in Electromagnetics," *Spreadsheets in Education (eJSiE)*: Vol. 4: Iss. 1, Article 1.
Available at: http://epublications.bond.edu.au/ejsie/vol4/iss1/1

This Regular Article is brought to you by the Faculty of Business at ePublications@bond. It has been accepted for inclusion in Spreadsheets in Education (eJSiE) by an authorized administrator of ePublications@bond. For more information, please contact Bond University's Repository Coordinator.

Spreadsheet Implementations for Solving Boundary-Value Problems in Electromagnetics

Abstract
Electromagnetics is arguably one of the most challenging courses in any electrical engineering curriculum. A solid foundation in vector calculus and a good intuition based on physical grounds are the normal requirements for a student to successfully complete this course. This paper presents a simple, yet powerful approach to introducing boundary-value problems arising in electrostatics. The principles of electrostatics find numerous applications such as electrostatic machines, lightning rods, gas purification, food purification, laser printers, and crop spraying, to name a few.

This paper focuses on the use of spreadsheets for solving electrostatic boundary-value problems. Sample problems that introduce the finite difference and the finite element methods are presented. The geometries included in the problems are sufficiently nontrivial for hand calculation or analytical solution, but reasonably manageable using spreadsheets. Although specialized software is available for this purpose, oftentimes such sophistication tends to obscure the mathematical underpinnings of the solution methods. Spreadsheets offer a transparent alternative – perhaps proximate to hand calculation – for students to better appreciate the numerical methods for solving boundary-value problems.

Keywords
Finite difference method, finite element method, Laplace's equation, boundary-value problems, electrostatics.

Cover Page Footnote
The authors gratefully acknowledge support from the Universidad del Turabo.

This regular article is available in Spreadsheets in Education (eJSiE): http://epublications.bond.edu.au/ejsie/vol4/iss1/1

1. Introduction

Many phenomena arising in science and engineering are modeled by partial differential equations (PDEs). In such cases the quantity of interest (e.g., temperature, potential, or displacement) is a function that depends on more than one variable (typically, space variables x, y, z and the temporal variable t). The heat equation, wave equation, and Laplace's equation are among the most prominent PDEs that undergraduate engineering students will encounter. The usual practice is to introduce the student to the analytical solution of these equations via the method of separation of variables. Under the assumption of linearity, the method naturally leads to the formulation of solutions as Fourier series expansions.

Treatment of PDEs and boundary-value problems (BVPs) may be found in many standard books [2 – 4, 7]. Reference 2 provides a very accessible presentation of the topic, while references 3, 4, and 7 provide a more concise presentation geared toward compendium courses in engineering mathematics. This paper will not expound the theories that provide the mathematical underpinnings of PDEs; instead, the paper emphasizes numerical solutions to PDEs and suggests implementations through spreadsheets.

This paper focuses on certain numerical methods for solving PDEs; in particular, the *finite difference* and the *finite element methods* are presented in the context of problems arising in electrostatics. Much of the development of these methods will follow those found in electromagnetics books [8]. The examples presented in this paper include geometries that are sufficiently nontrivial for hand calculation or analytical solution, but reasonably manageable by using spreadsheets. Although specialized software is available for this purpose, oftentimes such sophistication tends to obscure the inner workings of the numerical methods employed in the solution of PDEs. Spreadsheets offer a transparent alternative − perhaps proximate to hand calculation − for students to better appreciate the numerical methods for solving PDEs and BVPs.

The use of spreadsheets in teaching finite element analysis has been reported in the literature [9]. Reference 9 presents finite element analysis

in the context of a plane truss structure, wherein the objective is the determination of stresses given certain strains; the successful use of spreadsheets for introducing finite element analysis in an undergraduate mechanical engineering class is also discussed therein. A solution to the heat equation using spreadsheet models and 3D graphics is provided in [6]. Likewise, a menu-driven spreadsheet for solving electromagnetics problems can be found in [10]. In this paper, however, solutions to electromagnetics problems are presented in a manner that makes no use of macro programming; besides, this paper introduces finite element analysis via spreadsheets.

To the best of the authors' knowledge, spreadsheets are more prevalent in mechanical and civil engineering. The uses of spreadsheets in electrical engineering appear to be more limited – perhaps in this branch of engineering the use of specialized software is more consonant with the dynamic nature of the field. However, educators and students who wish to stress the importance of fundamentals, rather than effortless and expedite solutions, may want to consider using spreadsheets as a viable tool. Spreadsheets can offer a reasonable tradeoff between user-defined programming and specific-purpose software. It is in this spirit that this paper is presented, continuing the efforts initiated by the authors in a power systems course [5].

This paper is organized as follows. Section 2 presents the finite difference method for solving an electrostatic problem and includes the corresponding spreadsheet implementation. Section 3 presents the finite element method along with a spreadsheet implementation of the method. Section 4 compares the results obtained by each method. Section 5 discusses the pedagogical advantages of the spreadsheet implementations. Finally, Section 6 gives concluding remarks.

2. The finite difference method

The *finite difference method* (FDM) is conceptually simple. The problems to which the method applies are specified by a PDE, a solution region (geometry), and boundary conditions. Only a brief outline of the method

is given in this paper; for more detailed derivations the reader may consult [8]. The finite difference method entails three basic steps:

(1) Divide the solution region into a grid of nodes. Grid points are typically arranged in a rectangular array of nodes.

(2) Approximate the PDE and boundary conditions by a set of linear algebraic equations (the *finite difference equations*) on grid points within the solution region.

(3) Solve this set of linear algebraic equations.

The method is illustrated with an example arising in electrostatics. Consider the charge-free region depicted in Figure 1. The region has prescribed potentials along its boundaries. The region is divided into a rectangular grid of nodes, with the numbering of free nodes as indicated in the figure.

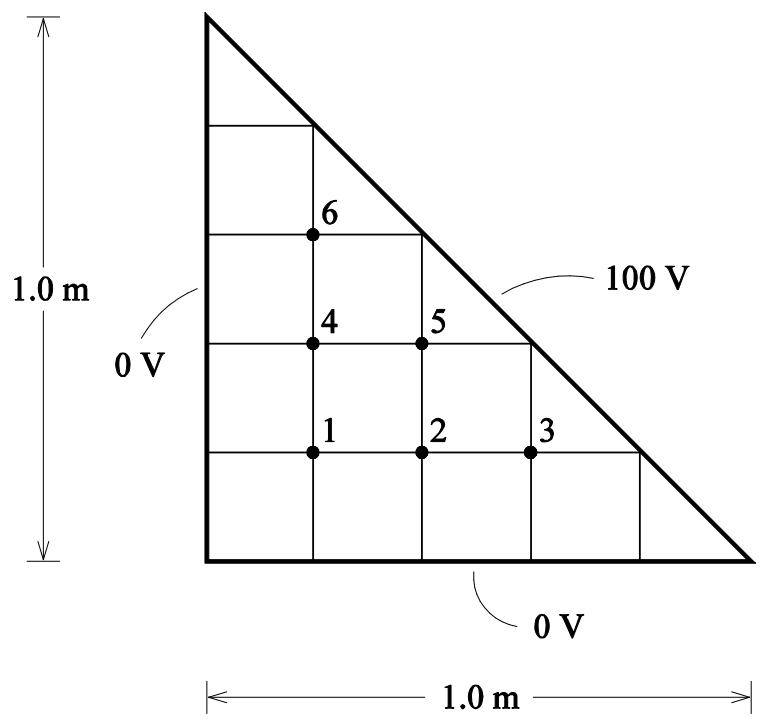

Figure 1: Charge-free region showing prescribed potentials at the boundaries and rectangular grid of free nodes to illustrate the finite difference method.

The potential $V = V(x,y)$ at an interior point (x,y) within the region is governed by the two-dimensional Laplace's equation

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0. \tag{1}$$

Let the location of an interior grid point be identified by a pair of integers (i,j), where i and j represent the position along the horizontal and vertical directions, respectively. For a grid having equal horizontal and vertical step sizes, the potential is given by the finite difference equation

$$V_{i,j} = \frac{1}{4}\left(V_{i+1,j} + V_{i-1,j} + V_{i,j+1} + V_{i,j-1}\right). \tag{2}$$

In essence, the potential at an interior grid point is the average of its four closest neighboring grid points that lie along the horizontal and vertical grid lines that intersect at the point. For example, the finite difference equation for node 4 in Figure 1 is

$$V_4 = \frac{1}{4}(V_5 + 0 + V_6 + V_1). \tag{3}$$

Similar equations are formulated for the remaining free nodes leading to a system of linear algebraic equations. This system of equations may be solved by a variety of methods. In this section the Gauss-Seidel method is implemented in a spreadsheet to solve this system of equations. The Gauss-Seidel method is a relatively simple iterative method for solving systems such as those encountered in the finite difference formulation.

The steps to implementing the spreadsheet for FDM are given below:

(1) The first step in the spreadsheet implementation is to input the prescribed potentials at the boundaries of the solution region. This step is shown in Figure 2. The prescribed potentials occupy the cell range C6:C8 and correspond to the potentials (0 V and 100 V) given in Figure 1. The user may change the numerical values of these potentials to accommodate different boundary conditions.

	A	B	C	D	E	F	G	H
1	Laplace Equation							
2	Geometry from Figure 15.32, page 707, Elements of Electromagnetics, 3rd edition,							
3	by Matthew Sadiku, Oxford University Press, 2001							
4								
5	Boundary conditions							
6	Left side		0	V				
7	Bottom side		0	V				
8	Slanted side		100	V				

Figure 2: Input section of spreadsheet implementation of the finite difference method.

(2) The next step is the implementation of the Gauss-Seidel method for solving the finite difference equations. There are 6 potentials at interior grid points that need to be determined (nodes 1 through 6 in Figure 1). These have been labeled V1 through V6 and occupy the cell range B14:G14 in Figure 3. To start the iterations initial estimates are given in B15:G15; in this case all initial estimates have been set to zero.

	A	B	C	D	E	F	G	H	I
12	Potentials at interior grid points								
13	Method 1: Finite difference, Gauss-Seidel iteration								
14	Iteration	V1	V2	V3	V4	V5	V6		
15	0	0.000	0.000	0.000	0.000	0.000	0.000	(Initial guesses)	
16	1	0.000	0.000	50.000	0.000	50.000	50.000		
17	2	0.000	25.000	56.250	25.000	62.500	56.250		
18	3	12.500	32.813	58.203	32.813	66.406	58.203		
19	4	16.406	35.254	58.813	35.254	67.627	58.813		
20	5	17.627	36.017	59.004	36.017	68.008	59.004		
21	6	18.008	36.255	59.064	36.255	68.128	59.064		
22	7	18.128	36.330	59.082	36.330	68.165	59.082		
23	8	18.165	36.353	59.088	36.353	68.177	59.088		
24	9	18.177	36.360	59.090	36.360	68.180	59.090		
25	10	18.180	36.363	59.091	36.363	68.181	59.091		

Figure 3: Screenshot showing the Gauss-Seidel iterations of the difference equations.

The cell range B16:G16 in Figure 3 implements the difference equations which are then solved using the Gauss-Seidel method. For example, for node 4, cell E16 contains the Microsoft Excel formula that implements Equation (3)

=SUM($C$6,F15,B16,G15)/4.

Observe that the formula in cell E16 makes absolute references to a node with prescribed potential; this prescribed potential must remain constant throughout the Gauss-Seidel iterations. Making the appropriate absolute references will prevent spurious results from occurring when formulas are copied to other cells.

(3) The final step is to reproduce the formulas in cell range B16:G16 by using the Copy command of Microsoft Excel. The formulas have been copied over the cell range B17:G25. It can be observed from Figure 3 that convergence is reached after 10 iterations for a precision index of 10^{-2}.

3. The finite element method

The *finite element method* (FEM) is a numerical technique for solving PDEs. FEM was originally applied to problems in structural mechanics. Unlike FDM, FEM is better suited for solution regions having irregularly shaped boundaries. The finite element analysis involves four basic steps:

(1) Divide the solution region into a finite number of *elements*. The most common elements have triangular or quadrilateral shapes. The collection of all elements should resemble the original region as closely as possible.

(2) Derive governing equations for a typical element. This step will determine the *element coefficient matrix*.

(3) Assemble all elements in the solution region to obtain the *global coefficient matrix*.

(4) Solve the resulting system of equations.

Again, the method is illustrated with an example. Consider the same charge-free region shown in Figure 1. The region is divided into 25 equal triangular elements as indicated in Figure 4. The elements are identified by encircled numbers 1 through 25. In this discretization there are 21 global nodes numbered 1 through 21. In the paragraphs that follow, a brief outline of FEM is provided; for detailed derivations the reader may consult [8].

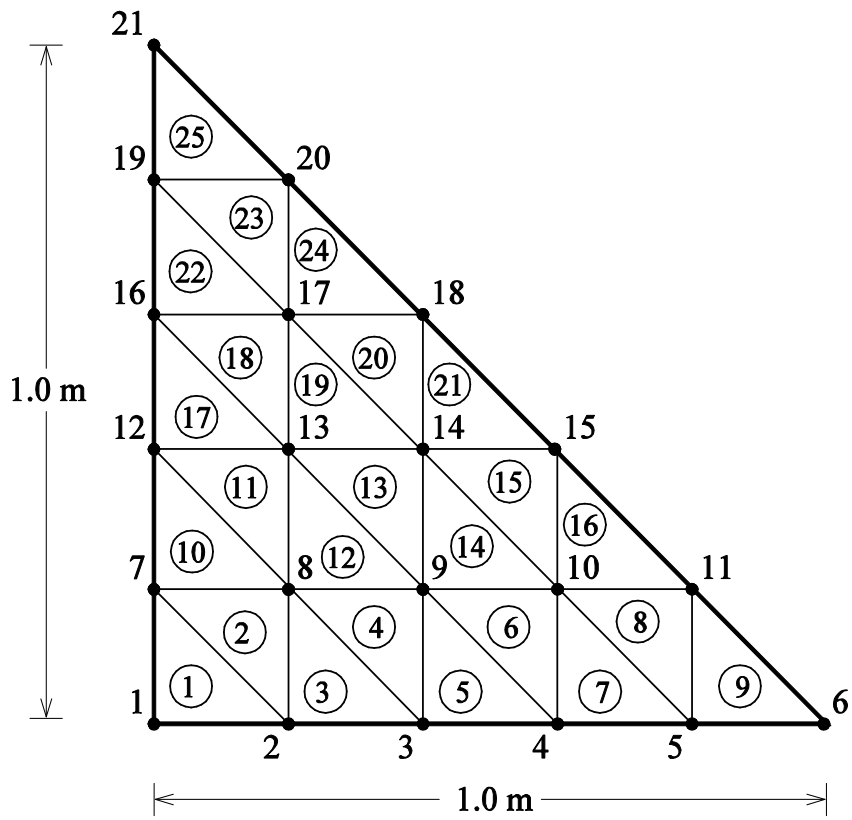

Figure 4: Finite element arrangement for electrostatic problem.

For each element e the following quantities are computed

$$\begin{aligned} P_1 &= y_2 - y_3, \quad P_2 = y_3 - y_1, \quad P_3 = y_1 - y_2, \\ Q_1 &= x_3 - x_2, \quad Q_2 = x_1 - x_3, \quad Q_3 = x_2 - x_1, \end{aligned} \tag{4}$$

where the subscripts refer to the local node numbers 1, 2, and 3 of element e. For example, in Figure 4, element 20 has global nodes 14, 18, and 17, which correspond, respectively, to local nodes 1, 2, and 3. Local node numbering is arbitrary; however, local node numbers must be assigned so that global nodes associated with an element are traversed in a counterclockwise sense. The coordinates (in meters) of nodes 14, 18, and 17 are (x_1,y_1) = (0.4, 0.4), (x_2,y_2) = (0.4, 0.6), and (x_3,y_3) = (0.2, 0.6), respectively. These sets of coordinates yield $P_1 = 0.0$, $P_2 = 0.2$, $P_3 = -0.2$, $Q_1 = -0.2$, $Q_2 = 0.2$, and $Q_3 = 0.0$.

With P_i and Q_i (i = 1, 2, 3) for element e thus computed, the entries of the 3 x 3 element coefficient matrix are then given by

$$C_{ij}^{(e)} = \frac{1}{4A}\left[P_iP_j + Q_iQ_j\right] \qquad (i,j = 1,2,3) \tag{5}$$

where

$$A = \frac{1}{2}\left[P_2Q_3 - P_3Q_2\right]. \tag{6}$$

As an example, for elements 20, 21 and 24, the element coefficient matrices computed according to Equations (5) and (6) are, respectively,

$$C^{(20)} = \begin{bmatrix} 0.5 & -0.5 & 0.0 \\ -0.5 & 1.0 & -0.5 \\ 0.0 & -0.5 & 0.5 \end{bmatrix}, \tag{7.a}$$

$$C^{(21)} = \begin{bmatrix} 1.0 & -0.5 & -0.5 \\ -0.5 & 0.5 & 0.0 \\ -0.5 & 0.0 & 0.5 \end{bmatrix}, \tag{7.b}$$

and

$$C^{(24)} = \begin{bmatrix} 1.0 & -0.5 & -0.5 \\ -0.5 & 0.5 & 0.0 \\ -0.5 & 0.0 & 0.5 \end{bmatrix}. \tag{7.c}$$

The global coefficient matrix is then assembled from the element coefficient matrices. Since there are 21 nodes, the global coefficient matrix will be a 21 x 21 matrix. In the following, the computations of one diagonal

and one off-diagonal entries are shown. For example, node 18, which corresponds to the $C_{18,18}$ entry in the global coefficient matrix C, belongs to elements 20, 21 and 24; since node 18 is assigned local node number 2 in elements 20 and 24, and local number 3 in element 21 (as seen in the middle table of Figure 5), the corresponding global coefficient is

$$C_{18,18} = C_{2,2}^{(20)} + C_{3,3}^{(21)} + C_{2,2}^{(24)} = 1.0 + 0.5 + 0.5 = 2.0. \tag{8}$$

Node potentials

Method 2: Finite elements

Nodal coordinates of finite element mesh

Node	x	y
1	0.0	0.0
2	0.2	0.0
3	0.4	0.0
4	0.6	0.0
5	0.8	0.0
6	1.0	0.0
7	0.0	0.2
8	0.2	0.2
9	0.4	0.2
10	0.6	0.2
11	0.8	0.2
12	0.0	0.4
13	0.2	0.4
14	0.4	0.4
15	0.6	0.4
16	0.0	0.6
17	0.2	0.6
18	0.4	0.6
19	0.0	0.8
20	0.2	0.8
21	0.0	1.0

Element-node identification

	Local Node		
Element	1	2	3
1	1	2	7
2	2	8	7
3	2	3	8
4	3	9	8
5	3	4	9
6	4	10	9
7	4	5	10
8	5	11	10
9	5	6	11
10	7	8	12
11	8	13	12
12	8	9	13
13	9	14	13
14	9	10	14
15	10	15	14
16	10	11	15
17	12	13	16
18	13	17	16
19	13	14	17
20	14	18	17
21	14	15	18
22	16	17	19
23	17	20	19
24	17	18	20
25	19	20	21

Prescribed potentials

Node	Fixed Potential
1	0.0
2	0.0
3	0.0
4	0.0
5	0.0
6	50.0
7	0.0
11	100.0
12	0.0
15	100.0
16	0.0
18	100.0
19	0.0
20	100.0
21	50.0

Figure 5: Input section of finite element analysis for electrostatic problem: (left) global node x and y coordinates; (middle) triangular element – global and local node correspondence; (right) nodes in the finite element mesh having prescribed potentials.

For the off-diagonal entry $C_{14,18}$, global link 14–18 corresponds to local link 1–2 of element 20 and local link 1–3 of element 21 and hence

$$C_{14,18} = C_{1,2}^{(20)} + C_{1,3}^{(21)} = -0.5 - 0.5 = -1.0. \tag{9}$$

Defining the vector of potentials v_f and v_p, where the subscripts f and p refer to nodes with free (unknown) potentials and prescribed potentials, respectively, the global coefficient matrix is then partitioned accordingly and the unknown potentials are obtained from

$$v_f = -C_{ff}^{-1} C_{fp} v_p. \tag{10}$$

The spreadsheet implementation of the finite element solution involves the following steps:

(1) Generate the input data section as shown in Figure 5. The input data consists of three tables: global node x and y coordinates; global and local node correspondence for each element; and list of nodes with prescribed (fixed) potentials.

(2) For each element, compute the values of P_i and Q_i from Equation (4) and obtain the element coefficient matrix from Equations (5) and (6), as shown in Figure 6. Notice that the rows for elements 4 through 23 were intentionally hidden so as to render a more manageable table.

	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z	AA	AB	AC
15	Element coefficient matrices																
16																	
17		Local	Global						SUM				SUM		Element		
18	Element	Node	Node	x	y	P1	P2	P3	Pi	Q1	Q2	Q3	Qi	A	Coefficient Mat		
19	1	1	1	0.0	0.0										1.00	-0.50	-0.50
20	1	2	2	0.2	0.0	-0.20	0.20	0.00	0.00	-0.20	0.00	0.20	0.00	0.02	-0.50	0.50	0.00
21	1	3	7	0.0	0.2										-0.50	0.00	0.50
22	2	1	2	0.2	0.0										0.50	-0.50	0.00
23	2	2	8	0.2	0.2	0.00	0.20	-0.20	0.00	-0.20	0.20	0.00	0.00	0.02	-0.50	1.00	-0.50
24	2	3	7	0.0	0.2										0.00	-0.50	0.50
25	3	1	2	0.2	0.0										1.00	-0.50	-0.50
26	3	2	3	0.4	0.0	-0.20	0.20	0.00	0.00	-0.20	0.00	0.20	0.00	0.02	-0.50	0.50	0.00
27	3	3	8	0.2	0.2										-0.50	0.00	0.50
88	24	1	17	0.2	0.6										1.00	-0.50	-0.50
89	24	2	18	0.4	0.6	-0.20	0.20	0.00	0.00	-0.20	0.00	0.20	0.00	0.02	-0.50	0.50	0.00
90	24	3	20	0.2	0.8										-0.50	0.00	0.50
91	25	1	19	0.0	0.8										1.00	-0.50	-0.50
92	25	2	20	0.2	0.8	-0.20	0.20	0.00	0.00	-0.20	0.00	0.20	0.00	0.02	-0.50	0.50	0.00
93	25	3	21	0.0	1.0										-0.50	0.00	0.50

Figure 6: Computation of element coefficient matrices $C^{(e)}$.

The table in Figure 6 is constructed as follows:

a. For element 1, global node information is linked to local node numbers. The link is implemented in the cell range O19:O21 via the Microsoft Excel VLOOKUP function. More precisely, cells O19, O20, and O21 contain, respectively, the formulas

```
=VLOOKUP($M19,$E$19:$H$43,2)
=VLOOKUP($M20,$E$19:$H$43,3)
=VLOOKUP($M21,$E$19:$H$43,4)
```

which search the item in column M (element number) and return the global nodes from the lookup table $E$19:$H$43.

b. The VLOOKUP function is invoked once again to retrieve global node coordinates. In the cell range P19:Q21 the following formulas implement this task:

```
=VLOOKUP($O19,$A$19:$C$39,2)      =VLOOKUP($O19,$A$19:$C$39,3)
=VLOOKUP($O20,$A$19:$C$39,2)      =VLOOKUP($O20,$A$19:$C$39,3)
=VLOOKUP($O21,$A$19:$C$39,2)      =VLOOKUP($O21,$A$19:$C$39,3)
```

c. Cell ranges R20:T20 and V20:X20 compute P_i and Q_i (i = 1, 2, 3) according to Equation (4) with the Excel formulas

=Q20-Q21	=Q21-Q19	=Q19-Q20
=P21-P20	=P19-P21	=P20-P19

d. Cells U20 and Y20 are optional. These cells simply compute $P_1 + P_2 + P_3$ and $Q_1 + Q_2 + Q_3$, as both sums must equal zero.

e. Cell Z20 implements Equation (6):

=(S20*X20-T20*W20)/2

f. The cell range AA19:AC21 computes the entries of the coefficient matrix for element 1 from Equation (5). For example, the entry $C_{12}^{(1)}$ is computed by the following Excel formula:

=(R20*S20+V20*W20)/(4*Z20)

g. The formulas in cell range N19:AC21 are copied over the cell range N22:AC93 to complete the computation of the remaining element coefficient matrices.

(3) The assembly of the global coefficient matrix is shown in Figure 7. The numbers in AF18:AZ18 and AE19:AE39 refer to global node numbers. The coefficients of the global matrix occupy cell range AF19:AZ39, which contain formulas similar to those in Equations (8) and (9). For example, the coefficient $C_{18,18}$ in cell AW36 contains the formula

=AB77+AC81+AB89,

while that of $C_{14,18}$ in cell AW32 is

=AB76+AC79.

Global coefficient matrix
(Nodes with prescribed potentials are indicated in red)

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21
1	1.0	-0.5	0.0	0.0	0.0	0.0	-0.5	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
2	-0.5	2.0	-0.5	0.0	0.0	0.0	0.0	-1.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
3	0.0	-0.5	2.0	-0.5	0.0	0.0	0.0	0.0	-1.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
4	0.0	0.0	-0.5	2.0	-0.5	0.0	0.0	0.0	0.0	-1.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
5	0.0	0.0	0.0	-0.5	2.0	-0.5	0.0	0.0	0.0	0.0	-1.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
6	0.0	0.0	0.0	0.0	-0.5	0.5	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
7	-0.5	0.0	0.0	0.0	0.0	0.0	2.0	-1.0	0.0	0.0	0.0	-0.5	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
8	0.0	-1.0	0.0	0.0	0.0	0.0	-1.0	4.0	-1.0	0.0	0.0	0.0	-1.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
9	0.0	0.0	-1.0	0.0	0.0	0.0	0.0	-1.0	4.0	-1.0	0.0	0.0	0.0	-1.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
10	0.0	0.0	0.0	-1.0	0.0	0.0	0.0	0.0	-1.0	4.0	-1.0	0.0	0.0	0.0	-1.0	0.0	0.0	0.0	0.0	0.0	0.0
11	0.0	0.0	0.0	0.0	-1.0	0.0	0.0	0.0	0.0	-1.0	2.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
12	0.0	0.0	0.0	0.0	0.0	0.0	-0.5	0.0	0.0	0.0	0.0	2.0	-1.0	0.0	0.0	-0.5	0.0	0.0	0.0	0.0	0.0
13	0.0	0.0	0.0	0.0	0.0	0.0	0.0	-1.0	0.0	0.0	0.0	-1.0	4.0	-1.0	0.0	0.0	-1.0	0.0	0.0	0.0	0.0
14	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	-1.0	0.0	0.0	0.0	-1.0	4.0	-1.0	0.0	0.0	-1.0	0.0	0.0	0.0
15	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	-1.0	0.0	0.0	0.0	-1.0	2.0	0.0	0.0	0.0	0.0	0.0	0.0
16	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	-0.5	0.0	0.0	0.0	2.0	-1.0	0.0	-0.5	0.0	0.0
17	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	-1.0	0.0	0.0	-1.0	4.0	-1.0	0.0	-1.0	0.0
18	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	-1.0	0.0	0.0	-1.0	2.0	0.0	0.0	0.0
19	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	-0.5	0.0	0.0	2.0	-1.0	-0.5
20	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	-1.0	0.0	-1.0	2.0	0.0
21	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	-0.5	0.0	0.5

Figure 7: Screenshot showing global coefficient matrix C.

(4) The matrices C_{ff} and C_{fp} are formed by extracting the appropriate rows and columns from the global coefficient matrix C. In this case, nodes 8, 9, 10, 13, 14, and 17 are the free nodes, while 1, 2, 3, 4, 5, 6, 7, 11, 12, 15, 16, 18, 19, 20, and 21 are the nodes with prescribed potentials. The result is shown in Figure 8.

(5) The final solution is obtained by using the matrix capabilities of Microsoft Excel. Although the matrix capabilities of Excel are fairly limited compared to those available in Matlab, Mathematica, or Maple, the ones provided by Excel are quite sufficient for the FEM implementation proposed in this paper. In particular, the spreadsheet functions MINVERSE (matrix inverse) and MMULT (matrix multiplication) are used in the implementation of Equation (10). This step is shown in Figure 9.

C_ff matrix (free nodes)

	8	9	10	13	14	17
8	4.00	-1.00	0.00	-1.00	0.00	0.00
9	-1.00	4.00	-1.00	0.00	-1.00	0.00
10	0.00	-1.00	4.00	0.00	0.00	0.00
13	-1.00	0.00	0.00	4.00	-1.00	-1.00
14	0.00	-1.00	0.00	-1.00	4.00	0.00
17	0.00	0.00	0.00	-1.00	0.00	4.00

C_fp matrix (free & prescribed nodes)

	1	2	3	4	5	6	7	11	12	15	16	18	19	20	21
8	0.00	-1.00	0.00	0.00	0.00	0.00	-1.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
9	0.00	0.00	-1.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
10	0.00	0.00	0.00	-1.00	0.00	0.00	0.00	-1.00	0.00	-1.00	0.00	0.00	0.00	0.00	0.00
13	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	-1.00	0.00	0.00	0.00	0.00	0.00	0.00
14	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	-1.00	0.00	-1.00	0.00	0.00	0.00
17	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	-1.00	-1.00	0.00	-1.00	0.00

Figure 8: Matrices C_{ff} and C_{fp} obtained from global coefficient matrix C.

Final calculations (band matrix method)

A^(-1) = C_ff^(-1)

0.2955	0.0909	0.0227	0.0909	0.0455	0.0227
0.0909	0.3152	0.0788	0.0485	0.0909	0.0121
0.0227	0.0788	0.2697	0.0121	0.0227	0.0030
0.0909	0.0485	0.0121	0.3152	0.0909	0.0788
0.0455	0.0909	0.0227	0.0909	0.2955	0.0227
0.0227	0.0121	0.0030	0.0788	0.0227	0.2697

b = -C_fp * Vp
(Vp = vector of prescribed potentials)

0.00
0.00
200.00
0.00
200.00
200.00

Vf = A^(-1) * b
(Vf = potential at free nodes)

Node	Potential
8	18.182
9	36.364
10	59.091
13	36.364
14	68.182
17	59.091

Figure 9: Final calculations section of finite element solution to electrostatic problem.

The implementation of Equation (10) has been broken down into three parts: (i) computation of the inverse of the C_{ff} matrix (this has been labeled $A^{-1} = C_{ff}^{-1}$ in Figure 9); (ii) computation of an intermediate vector $b = -C_{fp}v_p$; and (iii) computation of vector of potentials at free nodes $v_f = A^{-1}b$.

4. Discussion of results

In Sections 2 and 3 spreadsheet implementations of FDM and FEM were presented. As indicated in Table 1, the potentials at the free nodes computed by both methods compared fairly well. The node numbers in the table for FDM correspond to those in Figure 1, while those for FEM correspond to the node numbers shown in Figure 4. A better agreement can be obtained if more iterations of the Gauss-Seidel method are performed; this can be easily accommodated by copying a complete row of existing formulas, say cell range B25:G25 in Figure 3, to new rows until a desired level of agreement is achieved.

Table 1: Comparison of results obtained by FDM and FEM.

Finite difference		Finite element	
Node	Potential (V)	Node	Potential (V)
1	18.180	8	18.182
2	36.363	9	36.364
3	59.091	10	59.091
4	36.363	13	36.364
5	68.181	14	68.182
6	59.091	17	59.091

In general, FEM can better handle complex geometries and boundary conditions, while FDM is more suitable for solution regions having a certain degree of regularity. The tradeoff between the simplicity and the generality of the methods is evidenced by the amount of programming required by each method.

The potential distribution may be conveniently displayed graphically as depicted in Figure 10. The 2D chart on the left of the figure shows the potential distribution using conditional formatting for the color scheme, as demonstrated in [1]. The 3D chart on the right simply shows the potential distribution, with the heights of the cylinders being proportional to the value of the calculated potentials.

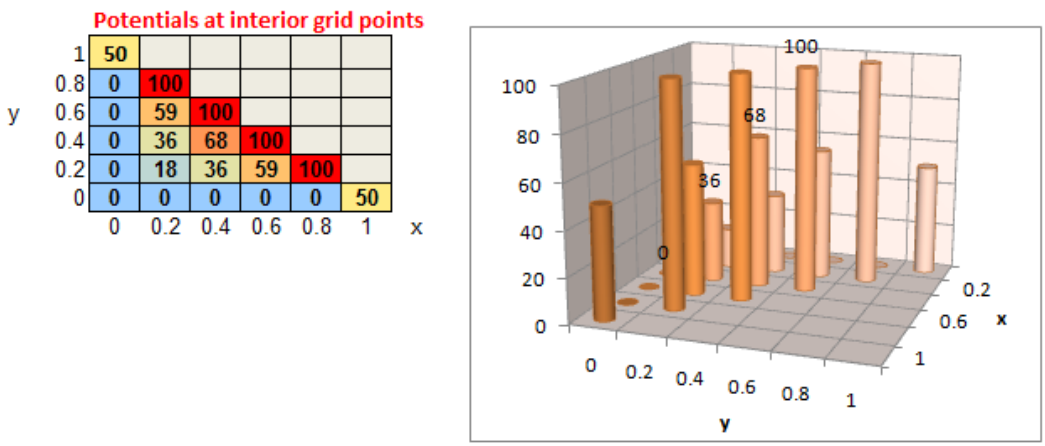

Figure 10: Graphical display of potential distribution.

5. Pedagogical advantages of spreadsheet implementations

In light of the spreadsheet implementations presented in this paper, the following remarks could be made:

(i) Setting up spreadsheets demands precise attention to detail from the user. Incorrect use of the `Copy` command or improper referencing of cells will lead to erroneous results. Attention to detail is a desirable skill that students should hone. Spreadsheets offer an environment in which such skill can be honed; the use of highly specialized software without proper understanding of the underlying methods may at times impede development of the skill in the student.

(ii) Spreadsheets offer a reasonable tradeoff between the sophistication of specific-purpose software and programming. The amount of programming in a typical spreadsheet is minimal, often reduced to formula editing and copying. This approach allows students to concentrate on analysis and interpretation of results rather than on time-consuming code debugging.

(iii) The spreadsheet implementations mimic hand calculations. The notepad-like interface of spreadsheets allows the student to keep track of results and ascertain convergence. Because of the resemblance to hand calculations, the spreadsheet approach may provide the student with a deeper understanding of the numerical methods, which could be obscured if specific-purpose software is used without proper knowledge of such methods.

(iv) The spreadsheets implementations may be presented to students to introduce numerical methods for solving BVPs. Students may be asked to implement similar spreadsheets to solve other types of PDEs and BVPs. The ambitious students may even improve upon the spreadsheets presented in this paper by macro programming or creative use of other Microsoft Excel functions.

In fairness to specific-purpose software designed for BVPs modeled by PDEs, it can be argued that such programs serve other purposeful needs, namely, handling large-scale systems, accommodating highly irregular geometries, and handling complex boundary conditions. Problems of considerable size and high complexity may not be handled efficiently by spreadsheets.

6. Conclusions

This paper presented spreadsheet implementations of two numerical methods for solving electrostatics problems. The spreadsheet approach is ideal if the emphasis is on understanding of numerical techniques. Spreadsheets may be considered as a viable alternative to enhancing education in other subjects and engineering fields.

The interested reader may obtain a copy of the Microsoft Excel file that implements the FDM and FEM solutions by sending an e-mail to mlau@suagm.edu.

References

[1] Abramovich, S. and Sugden, S. Spreadsheet conditional formatting: an untapped resource for mathematics education, *eJournal of Spreadsheets in Education*, Vol. 1, Issue 2, Article 3, 2004.
Online: http://epublications.bond.edu.au/ejsie/vol1/iss2/3.

[2] Asmar, N. *Partial differential equations with Fourier series and boundary value problems*, 2nd edition, Prentice Hall, 2004.

[3] Greenberg, M. *Advanced engineering mathematics*, 2nd edition, Prentice Hall, 1998.

[4] Kreyszig, E. *Advanced engineering mathematics*, 9th edition, John Wiley & Sons, 2006.

[5] Lau, M. A. and Kuruganty, S. P. Spreadsheet implementations for solving power-flow problems, *eJournal of Spreadsheets in Education*, 3(1): 27-45, August 2008. Online: http://epublications.bond.edu.au/ejsie/vol3/iss1/3.

[6] Neuwirth, E. and Arganbright, D. *The active modeler: mathematical modeling with Microsoft Excel*, 141-155, Brooks/Cole, 2004.

[7] O'Neil, P. *Advanced engineering mathematics*, 6th edition, CL-Engineering, 2006.

[8] Sadiku, M. *Elements of electromagnetics*, 4th edition, Oxford University Press, 2006.

[9] Teh, K. and Morgan, L. The application of Excel in teaching finite element analysis to final year engineering students, *Proceedings of the 2005 ASEE/AaeE (Australian Association for Engineering Education) Global Colloquium on Engineering Education*, Paper # 50.

[10] Yamani, A. and Kharab, A. Use of a spreadsheet program in electromagnetics, *IEEE Transactions on Education*, 44(3): 292-297, 2001.