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Insight into the Fractional Calculus via a Spreadsheet

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Abstract
Many students of calculus are not aware that the calculus they have learned is a special case (integer order) of fractional calculus. Fractional calculus is the study of arbitrary order derivatives and integrals and their applications. The article begins by stating a naive question from a student in a paper by Larson (1974) and establishes, for polynomials and exponential functions, that they can be deformed into their derivative using the μ-th order fractional derivatives for $0<\mu<1$. Through the power of Excel we illustrate the continuous deformations dynamically through conditional formatting. Some applications are discussed and a connection made to mathematics education.

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Keywords
Spreadsheet, fractional calculus, mathematics education

Insight into the Fractional Calculus via a Spreadsheet

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Abstract

Many students of calculus are not aware that the calculus they have learned is a special case (integer order) of fractional calculus. Fractional calculus is the study of arbitrary order derivatives and integrals and their applications. The article begins by stating a naive question from a student in a paper by Larson (1974) and establishes, for polynomials and exponential functions, that they can be deformed into their derivative using the μ-th order fractional derivatives for $0 < \mu < 1$. Through the power of Excel we illustrate the continuous deformations dynamically through conditional formatting. Some applications are discussed and a connection made to mathematics education.

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1 Introduction

In a 1974 paper, Larson relates a story that occurred in his calculus class. A student asked, after a class exercise that required students to sketch the derivative of a function by looking at the function's graph, "*If we sketched the half-derivative wouldn't its graph be about half way between the function and its first derivative*?" [11, p68].

Is the student right? It is moments like this in class that a professor might be caught off guard and think students do not understand. However, sometimes a seemingly naïve question from a student uncovers something hidden or previously unknown. The professor places his hand on his chin and thinks about the question as the students sit quietly. Finally, he responds, "*there is no such thing as the half-derivative of a function*". The class continues as the professor begins to talk about implicit differentiation, but the question still bounces around in his head. Does the half derivative of a function really not exist? Was I truthful with the students? The professor researches the concept and finds an answer to his questions.

2 Motivation and mathematics education

The initial motivation of this article is to illustrate dynamically, through modern technology, that graphs of fractional derivatives for certain functions are the intermediate graphs between the function and its first derivative. That is, develop dynamic graphs of the deformations in which the reader can visualize the deformations instead of static graphs as in Larson's original paper [11]. It may be regarded as analogous to how the non-integer real numbers fill in the number line between each of the integers. The authors also would like to express another motivation: to show how Excel can be used to illustrate advanced mathematics so that students that understand basic differentiation can begin to understand fractional derivatives of several basic functions. In addition, through Excel undergraduate students can build on their understanding of calculus to work on capstone type projects and enhance their learning of mathematics. Arganbright [1] states "*the availability of interactive and animated visualizations afforded by modern technology can enhance the learning process significantly..... At the same time, a spreadsheet, such as Microsoft Excel, provides us with a natural, interactive medium for doing mathematics. Perhaps surprisingly, spreadsheets are also effective tools for designing mathematical animations.*" For example, Excel is used to investigate the Goldbach conjecture by Baker *et al* [3]. Benacka [4] illustrates how to use it to produce 3D graphing, Wischniewsky [22] to create movie-like Lissajous animations, and Baker *et al* [2] to illustrate recursion.

2.1 Calculus reform

The teaching of calculus went through considerable change through the calculus reform movement, which stressed that calculus should be taught in a new way. It was one of the first movements that began to question how calculus had been taught for years in college and whether calculus was meeting the needs of its audience. The calculus reform movement, which began in the early 1980's, did not attempt to challenge the content of calculus, but to examine the way calculus was being taught [12]. The beginning of the movement was concerned with changing calculus from being a filter for the further study of math, science, and engineering, to being a pump for those fields. Long [12] stated that calculus was "the one class that seemed to be making or breaking students in mathematics and science" (p.3). The calculus reform movement tried to enhance the way students learned mathematics by using pedagogical techniques that were radically different from the norm of how calculus had been taught in the past [12]. One textbook which emerged from this movement was the Harvard calculus book [8]. Sher[20] stated that "the spreadsheet is the ideal environment for software that follows the Harvard approach." The Harvard calculus stressed that every topic should be presented geometrically, numerically, and algebraically, with a fourth – on the web – being added in recent years. Carefully crafted spreadsheets can be used to investigate topics geometrically and numerically to reinforce students algebraic work. Krantz states that "students might discuss and collaborate profitably if (computer-aided) material is put before them that will stimulate such interaction" ([9], p916); that will allow students to make mathematical discoveries when the designed activities lead him or her to it (*op cit*).

We seek to stimulate interest in the fractional calculus, and it is our hope that this article may introduce it to those unacquainted. The spreadsheet environment was found to be eminently suitable for the graphical illustration of the deformation of some simple classes of functions to their first derivatives.

3 A brief history of fractional calculus and basic definitions

The fractional derivative—the arbitrary order derivative—has a rich history dating back to around the time of the invention of calculus [13]. Many famous mathematicians and scientists have worked on fractional calculus. A useful, modern introduction, giving a brief historical overview, is that of Watson [21]. The present article considers some basic definitions of fractional derivatives and goes on to examine in more depth the answer to the student's question in terms of polynomials, exponential, sine and cosine functions. The reasons for including these functions and not others is that the authors want this article to be one that would draw readers into learning more on their own. Furthermore, the answers to the student question above yield easily accessible answers through Excel. For a more in-depth view of both fractional derivatives and fractional integrals see (Miller and Ross[13], Oldham[14], Post[18]).

The notation $D^\mu(f)$ (or simply $D^\mu f$) denotes the μ-th order fractional derivative of the function f where $\mu > 0$. This notation resembles the operator notation that students are shown in calculus. One of several fractional derivative definitions, found in the book *An Introduction to Fractional Calculus and Fractional Differential Equations* [13] by Miller and Ross, is derived using the Riemann-Liouville definition of the fractional integral (Miller and Ross first define the fractional integral and then the fractional derivative). To motivate the definition for the fractional derivative of a power function x^n where n is a positive integer and $x > 0$, we look at an observation of Lacroix in 1819. We know that for $n \geq m$ that:

$$\frac{d^m}{dx^m}(x^n) = n(n-1)(n-2)\dots(n-m+1)x^{n-m} \tag{1}$$

A general definition of the gamma function for complex z appears in eq (16). For all positive integers, we have $\Gamma(n) = (n-1)!$. Using this definition we can rewrite eq (1) as

$$D^m(x^n) = \frac{\Gamma(n+1)}{\Gamma(n-m+1)}x^{n-m} \tag{2}$$

Let n be any positive integer, $\mu > 0$, and $x \geq 0$. Then

$$D^\mu(x^n) = \frac{\Gamma(n+1)}{\Gamma(n-\mu+1)}x^{n-\mu} \qquad \text{(Lacroix)} \tag{3}$$

This definition, observed by Lacroix, is a natural extension of the power rule that students learn in calculus. A few years after Lacroix's definition, Fourier defined the fractional derivative of arbitrary order by using the Fourier transform of a function. Using modern day notation, he stated

$$D^{\mu} f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (i\alpha)^{\mu} \widehat{f}(\alpha) e^{ix\alpha} d\alpha$$

where $\widehat{f}$ stands for the Fourier transform of the function f. Liouville followed this with two definitions. For the first definition, Liouville stated that for any function $f(x)$ that can be extended into a series $\sum_{n=0}^{\infty} c_n e^{a_n x}$ where the real part of $a_n > 0$,

$$D^{\mu} f(x) = \sum_{n=0}^{\infty} c_n a_n^{\mu} e^{a_n x} \tag{4}$$

For the second definition

$$D^{\mu} x^{-a} = \frac{(-1)^{\mu} \Gamma(a+\mu)}{\Gamma(a)} x^{-a-\mu} \tag{5}$$

for $a > 0$ and $x > 0$. Although up to that time the definitions dealt with fractional derivatives, Riemann did not state a definition of the fractional derivative of $f(x)$. He defined the fractional integral of $f(x)$ to be

$$D^{-\mu} f(x) = \frac{1}{\Gamma(\mu)} \int_{c}^{x} (x-t)^{\mu-1} f(t)\ dt + \Psi(x)$$

where $\Psi(x)$ is called a complementary function and was included in the definition because of the ambiguity of the lower limit of integration. Sonin stated, cited in [13], in a paper in 1869 that

$$_{c}D_{x}^{-\mu} f(x) = \frac{1}{\Gamma(\mu)} \int_{c}^{x} (x-t)^{\mu-1} f(t)\ dt \tag{6}$$

where the real part of $\mu > 0$. Four more precise definitions emerged from these early formulas. The following four definitions stated in [13], respectively those of Riemann, Liouville, Riemann-Liouville, and Weyl, are consequences of (6) and are:

$$_{c}D_{x}^{-\mu} f(x) = \frac{1}{\Gamma(\mu)} \int_{c}^{x} (x-t)^{\mu-1} f(t)\ dt \qquad \text{(Riemann)} \tag{7}$$

where f satisfies $f(x) = O(x^{-1+\epsilon})$ for some $\epsilon > 0$,

$$_{-\infty}D_{x}^{-\mu} f(x) = \frac{1}{\Gamma(\mu)} \int_{-\infty}^{x} (x-t)^{\mu-1} f(t)\ dt \qquad \text{(Liouville)} \tag{8}$$

where f satisfies $f(x) = O(x^{-\mu-\epsilon})$ for $\epsilon > 0$, real part of μ is greater than 0, and $x \to \infty$,

$$_0D_x^{-\mu} f(x) = \frac{1}{\Gamma(\mu)} \int_0^x (x-t)^{\mu-1} f(t)\ dt \qquad \text{(Riemann-Liouville)} \tag{9}$$

where f satisfies $f(x) = O(x^{-\mu-\epsilon})$ for $\epsilon > 0$, real part of μ is greater than 0, and $x \to \infty$,

$$_xD_\infty^{-\mu} f(x) = \frac{1}{\Gamma(\mu)} \int_x^\infty (t-x)^{\mu-1} f(t)\ dt \qquad \text{(Weyl)} \tag{10}$$

In order to define the fractional derivative, Miller and Ross [13] first define the fractional integral. Suppose that the real part of μ is positive and let f be piecewise-continuous on $(0, \infty)$ and integrable on any finite subinterval of $[0, \infty)$. Then for $t > 0$ we denote that f is in class C if it satisfies the Riemann-Liouville definition of eq (9) above. Given a function f of class C, $\nu > 0$, and m the smallest integer that is greater than or equal to ν, then the fractional derivative of f of order ν is defined as

$$D^\nu f(x) = D^m[D^{-\mu} f(x)] \tag{11}$$

for $x > 0$ (if it exists), where $\mu = m - \nu$.

Oldham and Spanier [14] state two other definitions. The Grűnwald definition states that

$$_aD_x^\mu f(x) = \lim_{N\to\infty} \left[\frac{(\delta_N x)^{-\mu}}{\Gamma(-\mu)} \sum_{j=0}^{N-2} \frac{\Gamma(j-\mu)}{\Gamma(j+1)} f(x - j\delta_N x) \right] \tag{12}$$

where $\delta_N x = \frac{x-a}{N}$ and $a < x$. The other definition [14, page 54], motivated by Cauchy's integral formula, is

$$D^\mu f(z) = \frac{\Gamma(\mu+1)}{2\pi i} \int_C \frac{f(\zeta)}{(\zeta - z)^{\mu+1}} d\zeta \tag{13}$$

where μ is any real number not equal to a negative integer and thus $(\zeta - z)^{-\mu-1}$ does not have a pole at $\zeta = z$ but a branch point. The authors have stated the above definitions to convey that there are many different ways to define the fractional derivative (and others we have not stated; for example through Laplace transforms) and so care has to be taken when working with fractional calculus. As a consequence of (9), we have

$$_0D_t^{-\mu}(t^\lambda) = \frac{\Gamma(\lambda+1)}{\Gamma(\mu+\lambda+1)} t^{\mu+\lambda}. \tag{14}$$

where $\mu > 0$, $\lambda > -1$, and $t > 0$.

The observation that Lacroix made in 1819 matches the Riemann-Liouville definition of the μ-th fractional derivative of the power function x^n where $x \geq 0$ and n is a positive integer found in [13]. Lacroix's definition, eq (3), is derived using eq (9). As a consequence of Lacroix's definition, we can say that given a polynomial $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$, where a_i is a real number for each $i = 1, 2, 3, \ldots, n$, then

$$D^{\mu}(p(x)) = D^{\mu}\left(\sum_{i=0}^{n} a_i x^n\right) = \sum_{i=0}^{n} a_i D^{\mu}(x^i) = \sum_{i=0}^{n} a_i \frac{\Gamma(i+1)}{\Gamma(i-\mu+1)} x^{i-\mu} \qquad (15)$$

where $\Gamma(i-\mu+1)$ can be calculated for each i using the general definition. Recall that the gamma function is a holomorphic function defined everywhere on the complex plane, except at the negative integers and zero, by

$$\Gamma(z) = \int_0^{\infty} t^{z-1} e^{-t} dt \qquad (16)$$

It follows by integration by parts that $\Gamma(z+1) = z\Gamma(z)$. In general, the m-th order derivative of simple functions (power function x^a ($a > -1$), polynomials, exponential, and sines and cosines) deform to their $m+1$-th derivative via the μ-th order fractional derivatives ($m < \mu < m+1$). We will focus on the case $m = 0$. Cases for any other integer m would follow in a very similar way. In fact, a translation back to the case $0 < \mu < 1$ can be made by letting $D^0 g = D^m f$ (so $D^1 g = D^{m+1} f$). To illustrate that these simple functions deform to their derivatives through μ-th fractional derivatives ($0 < \mu < 1$) the authors use Microsoft Excel 2007.

4 Continuous deformation of a polynomial into its derivative

We will see that if we take a polynomial $p(x)$ then the graphs of $D^{\mu}(p(x))$ for $0 < \mu < 1$ are intermediate graphs in the continuous deformation from the graph of the polynomial $p(x)$ to the graph of the derivative $p'(x)$. Some brief examples are considered which illustrate formula (3). These are the constant function $f(x) = 1$ (easiest example), the monomial second degree polynomial $g(x) = x^2$ (a fairly easy example), and a trinomial third degree polynomial $h(x) = (x-1)(x-2)(x-3)$ (a harder example).

Example 1 *Using eq (3) for the constant function* $f(x) = 1$, *we have the* μ*-th order fractional derivative for* $0 < \mu < 1$ *is*

$$D^{\mu}(f(x)) = D^{\mu}(1) = \frac{\Gamma(1)}{\Gamma(1-\mu)} x^{-\mu}.$$

The fractional derivative is not equal to zero for all $0 < \mu < 1$, but we see that this agrees with the statement that *the derivative of a constant is zero* from calculus since the gamma function has a pole at 0. That is, as $\mu \to 1$, $\Gamma(1-\mu) \to \infty$ and hence $\frac{\Gamma(1)}{\Gamma(1-\mu)} x^{-\mu} \to 0$.

Example 2 *Using eq (3) for the power function* $g(x) = x^2$ *then we know* $g'(x) = 2x$ *and using the definition above we have*

$$D^{\frac{1}{2}}(g(x)) = D^{\frac{1}{2}}(x^2) = \frac{\Gamma(2+1)}{\Gamma\left(2-\frac{1}{2}+1\right)}x^{2-\frac{1}{2}} = \frac{2!}{\Gamma\left(\frac{5}{2}\right)}x^{\frac{3}{2}} = \frac{2}{\frac{3}{4}\sqrt{\pi}}x^{\frac{3}{2}} = \frac{8}{3\sqrt{\pi}}x^{\frac{3}{2}} \tag{17}$$

using the fact that $\Gamma(n+1) = n!$, $\Gamma(z+1) = z\Gamma(z)$, and $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$. We will look at more of the continuous deformations in the later sections.

Example 3 *Using eq (15) for this next example we will look at the cubic polynomial*

$$p(x) = (x-1)(x-2)(x-3) = x^3 - 6x^2 + 11x - 6.$$

So the derivative is $p'(x) = 3x^2 - 12x + 11$ and the one-third derivative is

$$\begin{aligned}
D^{\frac{1}{3}}(p(x)) &= D^{\frac{1}{3}}(x^3 - 6x^2 + 11x - 6) \\
&= D^{\frac{1}{3}}(x^3) - 6D^{\frac{1}{3}}(x^2) + 11D^{\frac{1}{3}}(x) - 6D^{\frac{1}{3}}(1) \\
&= \frac{\Gamma(4)}{\Gamma\left(3-\frac{1}{3}+1\right)}x^{3-\frac{1}{3}} - 6\frac{\Gamma(3)}{\Gamma\left(2-\frac{1}{3}+1\right)}x^{2-\frac{1}{3}} \\
&\quad +11\frac{\Gamma(2)}{\Gamma\left(1-\frac{1}{3}+1\right)}x^{1-\frac{1}{3}} - 6\frac{\Gamma(1)}{\Gamma\left(0-\frac{1}{3}+1\right)} \\
&= \frac{\Gamma(4)}{\Gamma\left(\frac{11}{3}\right)}x^{\frac{8}{3}} - 6\frac{\Gamma(3)}{\Gamma\left(\frac{8}{3}\right)}x^{\frac{5}{3}} + 11\frac{\Gamma(2)}{\Gamma\left(\frac{5}{3}\right)}x^{\frac{2}{3}} - 6\frac{\Gamma(1)}{\Gamma\left(\frac{2}{3}\right)} \\
&= \frac{\Gamma(4)}{\left(\frac{8}{3}\right)\left(\frac{5}{3}\right)\left(\frac{2}{3}\right)\Gamma\left(\frac{2}{3}\right)}x^{\frac{8}{3}} - 6\frac{\Gamma(3)}{\left(\frac{5}{3}\right)\left(\frac{2}{3}\right)\Gamma\left(\frac{2}{3}\right)}x^{\frac{5}{3}} + 11\frac{\Gamma(2)}{\left(\frac{2}{3}\right)\Gamma\left(\frac{2}{3}\right)}x^{\frac{2}{3}} - 6\frac{\Gamma(1)}{\Gamma\left(\frac{2}{3}\right)} \\
&= \frac{1}{\Gamma\left(\frac{2}{3}\right)}\left(\frac{80}{41}x^{\frac{8}{3}} - \frac{54}{5}x^{\frac{5}{3}} + \frac{33}{2}x^{\frac{2}{3}} - 6\right)
\end{aligned}$$

These examples illustrate a few of the infinitely many fractional derivatives of a given function between $D^0 f$ and $D^1 f$. One may obtain a glimpse of the continuum of fractional derivatives by plotting a discrete number of them between $D^0 f$ and $D^1 f$ in Excel. A striking picture is obtained. Therefore, we see that there is an answer to the naive question that the student asked about the graph of the half-derivative of a function being half way between the graph of the function and its derivative, when we look at polynomials. That is, there is a continuous deformation of a polynomial into its derivative via fractional (arbitrary order) derivatives between 0 and 1. The graph of the fractional derivative is not always between the function and derivative, however, there is the continuous deformation. We will now examine the fractional derivatives of $e^{\sigma x}$ for different values of σ.

5 Continuous deformation of $e^{\sigma x}$ into its derivative

5.1 Simple case when $\sigma > 0$ is a real number

For the exponential function $e^{\sigma x}$ the μ-th order fractional derivative, where μ is any real number, is

$$D^{\mu}(e^{\sigma x}) = \sigma^{\mu} e^{\sigma x}. \tag{18}$$

This definition is derived using (11) and (8). We will only look at the μ-th order fractional derivatives for $0 < \mu < 1$. The examples below illustrate the deformations for $0 < \mu < 1$.

Example 4 *Let* $f(x) = e^{2x}$ *. Then* $D^{\mu}(e^{2x}) = 2^{\mu} e^{2x}$*. We see that* $D^0(f(x)) = f(x) \leq D^{\mu}(f(x)) \leq D^1(f(x))$*.*

Example 5 *Let* $f(x) = e^{\frac{1}{3}x}$*. Then* $D^{\mu}(e^{\frac{1}{3}x}) = (\frac{1}{3})^{\mu} e^{\frac{1}{3}x}$*. We see that* $D^1(f(x)) \leq D^{\mu}(f(x)) \leq D^0(f(x))$*.*

5.2 Case when $\sigma < 0$ is a real number

The calculation in the case of $e^{\sigma x}$, where $\sigma < 0$, is a little more complicated. Let $\sigma = -\rho$ where $\rho > 0$. Then

$$\begin{aligned} D^{\mu}(e^{\sigma x}) &= D^{\mu}(e^{-\rho x}) = (-\rho)^{\mu} e^{-\rho x} = (-1)^{\mu} \rho^{\mu} e^{-\rho x} \\ &= \rho^{\mu}(\cos(\mu\pi(2n+1)) + i\sin(\mu\pi(2n+1)))e^{-\rho x}. \end{aligned}$$

Note that the imaginary part tends to 0 as μ tends to 1 and the real part tends to $-\rho e^{-\rho x}$ as μ tends to 0.

5.3 Case when σ is a complex number

Let $f(x) = e^{\sigma x}$ where $\sigma = \varsigma + i\omega$. Writing σ in polar coordinates, we have $\sigma = \rho e^{i\theta}$. Therefore

$$f(x) = e^{x\rho e^{i\theta}} = e^{x\rho}(\cos\theta + i\sin\theta) = e^{x\rho\cos\theta}\left(\cos(x\rho\sin\theta) + i\sin(x\rho\sin\theta)\right)$$

and the real part of $f(x)$ is

$$e^{x\rho\cos\theta}\cos(x\rho\sin\theta)$$

and the imaginary part of $f(x)$ is

$$e^{x\rho\cos\theta}\sin(x\rho\sin\theta).$$

In addition, the derivative

$$D^1(f(x)) = \sigma e^{\sigma x} = (\varsigma + i\omega)e^{(\varsigma+i\omega)x} = \rho(\cos\theta + i\sin\theta)[\operatorname{Re} D^0 f(x) + i\operatorname{Im} D^0 f(x)].$$

Therefore the real part of $D^1 f(x)$ of the above expression is

$$\begin{aligned}\mathrm{Re}(D^1 f(x)) &= (\cos\theta\cos(x\rho\sin\theta) - \sin\theta\sin(x\rho\sin\theta))\rho e^{x\rho\cos\theta} \\ &= \cos(\theta + x\rho\sin\theta)\rho e^{x\rho\cos\theta}\end{aligned}$$

and similarly the imaginary part is

$$\mathrm{Im}(D^1 f(x)) = \sin(\theta + x\rho\sin\theta)\rho e^{x\rho\cos\theta}$$

Hence

$$\begin{aligned}D^1 f(x) &= \cos(\theta + x\rho\sin\theta)\rho e^{x\rho\cos\theta} + i\sin(\theta + x\rho\sin\theta)\rho e^{x\rho\cos\theta} \\ &= e^{i(\theta + x\rho\sin\theta)}\rho e^{x\rho\cos\theta} \\ &= \rho e^{i\theta} e^{ix\rho e^{i\theta}} \\ &= \sigma e^{\sigma x}.\end{aligned}$$

The formula for the fractional derivatives of $\sin x$ and $\cos x$ are included in the calculation of the fractional derivative of e^{ix} using the fact that $e^{ix} = \cos x + i\sin x$. That is,

$$\begin{aligned}& D^\mu(\cos x) + iD^\mu(\sin x) \\ =\ & D^\mu(\cos x + i\sin x) \\ =\ & D^\mu(e^{ix}) \\ =\ & i^\mu e^{ix} \\ =\ & e^{\mu\ln i} e^{ix} \\ =\ & e^{\mu\ln e^{i\pi/2}} e^{ix} \\ =\ & e^{\mu i\pi/2} e^{ix} \\ =\ & e^{i(x+\mu\pi/2)} \\ =\ & \cos(x + \mu\pi/2) + i\sin(x + \mu\pi/2)\end{aligned}$$

From this we conclude that $D^\mu(\cos x) = \cos(x + \mu\pi/2)$ and $D^\mu(\sin x) = \sin(x + \mu\pi/2)$. Thus, for this particular function, e^{ix}, we have a beautiful illustration of the fractional derivative as a rotational transformation, or phase shift. When μ takes integer values, these results reduce to the integer order derivatives from ordinary calculus by application of the formulas for sine and cosine of sums of angles from elementary trigonometry.

6 Excel models

6.1 Polynomial

We wish to illustrate the transition from function to first derivative via fractional derivatives in Excel. Consider the polynomial

$$p(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0 \tag{19}$$

We define

$$\alpha_i = \frac{\Gamma(i+1)}{\Gamma(i-\mu+1)} \tag{20}$$

so that

$$\frac{\alpha_i}{\alpha_{i-1}} = \frac{i}{i-\mu} \tag{21}$$

We have then

$$\begin{aligned}
\alpha_1 &= \frac{\Gamma(2)}{\Gamma(2-\mu)} = \frac{1!}{\Gamma(2-\mu)} \\
\alpha_2 &= \frac{\Gamma(3)}{\Gamma(3-\mu)} = \frac{2!}{(2-\mu)\,\Gamma(2-\mu)} \\
\alpha_3 &= \frac{\Gamma(4)}{\Gamma(4-\mu)} = \frac{3!}{(3-\mu)(2-\mu)\,\Gamma(2-\mu)}
\end{aligned}$$

etc.
This leads to:

$$\alpha_i = \frac{i!}{\Gamma(2-\mu)\prod_{j=1}^{i}(j-\mu)} \qquad \text{for} \quad 1 \le i \le n \tag{22}$$

Now write

$$\beta_i = \frac{\alpha_i}{\alpha_1} \qquad \text{for} \quad 1 \le i \le n$$

This gives us easily-computable betas, and the troublesome $\Gamma(2-\mu)$ may be factorized outside of the sum for the fractional derivative. The sequence (β_i) satisfies the simple recurrence

$$\beta_i = \frac{i\beta_{i-1}}{i-\mu} \qquad \text{for} \quad 2 \le i \le n \tag{23}$$

with $\beta_1 = 1$.
We now have:

$$\begin{aligned}
D^\mu(p(x)) &= D^\mu\left(\sum_{i=0}^{n} a_i x^i\right) = \sum_{i=0}^{n} a_i D^\mu\left(x^i\right) \qquad (24) \\
&= \sum_{i=0}^{n} a_i \frac{\Gamma(i+1)}{\Gamma(i-\mu+1)} x^{i-\mu} \\
&= a_0 \frac{\Gamma(1)}{\Gamma(1-\mu)} x^{-\mu} + \sum_{i=1}^{n} a_i \frac{\Gamma(i+1)}{\Gamma(i-\mu+1)} x^{i-\mu} \\
&= \frac{a_0(1-\mu)}{\Gamma(2-\mu)} x^{-\mu} + \frac{x^{-\mu}}{\Gamma(2-\mu)} \sum_{i=1}^{n} a_i \beta_i x^i \\
&= \frac{x^{-\mu}}{\Gamma(2-\mu)} \left(a_0(1-\mu) + \sum_{i=1}^{n} a_i \beta_i x^i\right) \qquad (25)
\end{aligned}$$

For the Excel model, we consider polynomials up to degree 5. The model makes use of the quantity `factor`. This is the factor outside the large parentheses just above, in eq (25). To render it in Excel, its logarithm is computed and then the exponential function invoked. This is done in column H.

We now briefly describe how the spreadsheet computes the quantity in eq (25), given values of x, μ and the coefficients of the polynomial for $n = 5$. Horner's recursive method is used to evaluate the polynomial. Column B initializes the polynomial sum to a_5. Columns C through F progressively accumulate the sum of eq (25), while column G does the final multiplication by x and then adds in the term $a_0 (1 - \mu)$.

Column I performs a simple multiplication to compute the entire $D^{\mu} (p (x))$ as expressed by the right side of eq (25). The small block of parameters `M1:P2` is used for plotting the results on $a \leq x \leq b$ using $n + 1$ sample points. There is a small issue when $x = 0$ as the `factor` of eq (25) then becomes $0^{-\mu}$. Thus, we choose a plotting interval (a, b) where $a > 0$. The animation is achieved by a small amount of VBA code to drive changes in the slider-generated value of μ.

6.2 Excel model for the exponential cases

The function we consider is $f (x) = e^{\sigma x} = e^{(\varsigma + i\omega)x}$. Sliders (scrollbars) are introduced to allow the user to easily alter parameters. A slider is included for each of the parameters μ, ς, ω. In a very real sense, the Excel model is simpler than that for the polynomial case, since no gamma function is required. However, since we have a complex-valued function of a real variable, the real and imaginary parts of the fractional derivatives are displayed on separate charts. By setting either ς or ω to zero, we may observe the transition of pure exponential or sinusoidal functions respectively via fractional derivatives to their full first derivatives.

6.3 Excel model for the natural logarithm

For this we need some way to compute the *digamma function*, i.e., the logarithmic derivative of the gamma function. The book [14] gives the following formula for the fractional derivative of the natural logarithm.

$$D^{\mu} (\ln x) = \frac{x^{-\mu}}{\Gamma (1 - \mu)} (\ln x - \gamma - \psi (1 - \mu)) \tag{26}$$

where $\mu > 0$.

In eq 26, ψ is the *digamma function*, defined to be the logarithmic derivative of the gamma function, i.e.,

$$\psi (z) = \frac{\Gamma' (z)}{\Gamma (z)} \tag{27}$$

The number $\gamma = -\psi(1) \simeq 0.577215\,6649$ is Euler's constant. In order to implement eq 26 in Excel, we need a suitable means of computing a good approximation to the digamma function. A useful online source is [23].

7 Continuous deformation of a function into its antiderivative

Equations 14, 18, and 26 also hold for $\mu < 0$. That is, the fractional derivatives and fractional integrals for t^λ, $e^{\sigma x}$, and $\ln x$, can be stated concisely using (14), (18), and (26). This time the deformation is from the function to its antiderivative and the Excel models for the derivatives can be modified to show these cases by letting $-1 < \mu < 0$. In each of the models, the user may observe these deformations by using the sliders, or by clicking on the *Animate* button. Note that *macros* need to be enabled in order to use this functionality.

8 Some applications of fractional calculus

Although the fractional calculus had its genesis in the 1600s and has remained dormant until roughly the 1980s, it has been found to have numerous applications. Abel wrote about probably the first application of fractional calculus called the *tautochrone problem* [6, p279]. This is the problem of determining the shape of the curve such that the time of descent of a frictionless point mass sliding down the curve under the action of gravity is independent of the starting point.

We give just a brief overview of applications here and refer the interested reader to the website of Podlubny [17] for a very useful set of links for fractional calculus resources. It should not be thought that fractional calculus is merely a mathematical curiosity. Indeed, Kulish [10] describes application of fractional calculus to the solution of time-dependent, viscous-diffusion fluid mechanics problems. He states that "the fractional methodology is validated and shown to be much simpler and more powerful than existing techniques." A recent paper giving a very good overview of applications in engineering is that of Diethelm *et al* [7]. This work contains a very useful bibliography and much summary material that is difficult to find elsewhere.

Applications of fractional calculus have blossomed in the past 30 years and we make no attempt to present a complete list of applications. The purpose of such a list is to give the reader a basic idea of some of the applications so that they can explore the ones of interest. Spanier and Oldham [14] mention several applications in their book on diffusive transport in a semi-infinite medium including the following: heat in solids, chemical species in homogeneous media, vorticity in fluids, and electricity in resistive-capacitive lines. The book *Functional fractional calculus: For system identification and controls*, by Das [5] covers many applications in engineering and science. The work by Sabatier *et al* [19] describes a variety of applications in physics and engineering and states "we believe that researchers, new or old, would realize that we cannot remain within the boundaries of integer order calculus, that fractional calculus is indeed a viable mathematical tool that will accomplish far more than what integer calculus promises, and that fractional calculus is the calculus for the future.” [19, pp. xii].

9 Conclusion

We hope that we have gone some way to show the power and utility of the (misnamed) "fractional calculus". There is no doubt that this branch of mathematics will assume a much greater importance in the years to come. From a mathematics education point of view, it is also our hope that the Excel models which accompany this article will go some way to illustrate some of the relationships of the fractional calculus. In particular, we wish to highlight the following points:

1. That students may gain a renewed respect for the beauty, elegance and power of mathematics. For one unfamiliar with even its existence, to see that the fractional calculus is a generalization of standard calculus will often be a source of amazement. It may be compared to the experience of a student learning about the generalization of the factorial of a positive integer to the gamma function or the theory of groups being an abstraction of the overall structure of number systems with an associative binary operation.

2. The almost limitless vista of mathematics. As a student, to discern that the new branch of math that you have just learned is a generalization of an existing, familiar branch, should be a humbling experience: "the more mathematics you know the less you know", because it begins to dawn on you that there is so much mathematics that you do not know.

3. The possibilities for research, i.e., that students can work through interesting research projects to learn more about the history of calculus and fractional calculus in general, and thus significantly broaden their mathematical experiences.

4. That students can become more mathematically mature as they study more mathematics: learning the generalizations of calculus will in turn help students gain better understanding of calculus and push their learning beyond what they have learned before.

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