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Spreadsheets and the discovery of new knowledge

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Abstract
The paper introduces a new class of polynomials discovered as an extended inquiry into a two-parametric difference equation using a spreadsheet. These polynomials possess a number of interesting properties connected to the notion of a generalized golden ratio and can be used as a background for a spreadsheet-enhanced teaching of mathematics. The paper reflects on activities designed for a technology-rich mathematics education course for prospective teachers of secondary mathematics.

Keywords
spreadsheets, Fibonacci numbers, Fibonacci-like polynomials, difference equations, cycles

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Abstract

This paper shows how education-oriented spreadsheet-based explorations with Fibonacci numbers can result in the discovery of cycles of different periods formed by the orbits of a two-parametric difference equation of the second order. This equation is motivated through the introduction of the so-called Fibonacci sieve. The occurrence of the cycles is interpreted in terms of Fibonacci-like polynomials brought into being in the context of these explorations. This new class of polynomials possesses a number of interesting properties connected to the notion of a generalized golden ratio and can be used as a background for a spreadsheet-enhanced teaching of combinatorial identities and their formal demonstration. The paper reflects on activities designed for a technology-rich mathematics education course for prospective teachers of secondary mathematics. It is argued that an appropriate experience with a mathematical frontier can motivate the teachers to teach through a guided discovery mode.

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Key words: Spreadsheet, Fibonacci numbers, Fibonacci-like polynomials, difference equations, cycles, generalized Golden Ratio.

1 Introduction

This paper familiarizes the readers of the journal with a mathematical concept, called Fibonacci-like polynomials [1], brought into being in the context of education as the authors continued a collaboration on the development of technology-motivated methods of teaching topics in discrete mathematics at the collegiate level reflected earlier in [2]. In fact, it is through spreadsheet modeling, that the authors first discovered certain phenomena of modern mathematics in a classic linear discrete dynamical system of the second order originally studied by the outstanding German mathematician Oscar Perron

[14]. More specifically, unsophisticated spreadsheet-based explorations with Fibonacci numbers resulted in the discovery of cycles of different periods formed by the orbits generated for certain values of parameters of this system. The interpretation of the occurrence of the modern effects in the classic system led the authors to these polynomials. Such an unexpected discovery of the coexistence of modern and classic ideas that happened outside pure mathematical research underscored the mathematical, computational, and pedagogical power of an electronic spreadsheet and served as a motivation for writing this article for the journal specifically devoted to broad educational applications of the software.

In what follows, the authors will demonstrate how using a spreadsheet as an exploratory tool, one can go far beyond traditional modeling of Fibonacci numbers and the Golden Ratio, enabling several generalizations of these well-known concepts using the power of the numerical approach. The importance of the motivation of mathematical concepts by concrete examples in the teaching of mathematics stems from the commonly accepted notion that, nowadays, students are interested in the study of the subject matter if they are confident in the applicability of the material they are about to learn. This points at the educational value of concrete numeric problems as tools that enable one to approach mathematics "at least initially . . . from an experientially based direction, rather than an abstract/deductive one" [5, p96]. This paper will show how spreadsheet-based computational experiments provide learners of mathematics with experience in observations ("quasi-experiments," as Euler [6] called them) and their subsequent interpretation in the language of formal mathematics. The need to support "quasi-experiments" by mathematically rigorous reasoning was expressed by Euler in the following words: "Therefore, we should take great care not to accept as true such properties of numbers which we have discovered by observation and which are supported by induction alone. Indeed, we should use such a discovery as an opportunity to investigate more exactly the properties discovered and to prove or disprove them; in both cases we may learn something useful" (translated from Latin by Polya [15, p3]).

The material of this paper was used in part in a capstone course taught by one of the authors to prospective teachers of secondary mathematics. One of the recommendations of the Conference Board of the Mathematical Sciences [5] for the preparation of the teachers includes the need for "courses that develop deep understanding of mathematics they will teach" (p. 7) and "what it means to write a formal proof" (p. 14). Topics included in such courses should demonstrate that just like the complexity of a symphony stems gradually from an unpretentious tune, the structure of mathematics develops through numerous generalizations and abstractions of a simple idea. This reflects an intrinsic nature of a mathematical concept that arises from an intuitive idea to be used further in developing more and more abstract structures. It has been observed through the teaching of the capstone course that in spreadsheet-supported learning environments the teachers enjoy dealing with mathematical formalism and develop proficiency in writing proofs. Indeed, "designing a useful spreadsheet requires flexible ability to express numerical relationships in algebraic notation" [5, pp 127-128], and encourages the emergence of "deep questions [by the teachers] about the appropriate role of skill in algebraic manipulation" [5, p124]. This explains the authors' choice to combine spreadsheet explorations with formal mathematical demonstrations. In this paper, the focus is on difference equations – mathematical models of discrete dynamical systems found in applications to radio engineering, communication, and computer architecture research [10]. The methods of teaching introduced below will take advantage of the remarkable facility of spreadsheets to numerically model these equations. Furthermore, it will be shown that the possibility of the geometrization of algebraic proof techniques using the two-dimensional structure of spreadsheets can facilitate the comprehension of these techniques by providing a setting for a semi-formal manipulation of symbols as a support system for formal reasoning. Finally, the paper will demonstrate how a natural generalization and open-ended exploration of familiar models, made possible by the use of spreadsheets, can lead motivated learners of mathematics to a mathematical frontier.

2 Fibonacci and Lucas numbers

Consider the number sequence

$$1, 1, 2, 3, 5, 8, 13, 21, 34 \ldots \tag{1}$$

in which every number beginning from the third is the sum of the two preceding numbers and the first two numbers are equal to one. As is well known, these numbers are named after the 13th century Italian mathematician Fibonacci, who introduced them into Western mathematics by exploring the growth of the population of rabbits breeding in ideal circumstances. Changing one of the first two Fibonacci numbers (or both) and keeping its (recursive) definition without change, yields a new number sequence that is commonly referred to as a Fibonacci-like sequence. The most famous example of that kind is the sequence of Lucas numbers

$$2, 1, 3, 4, 7, 11, 18, 29, 47, \ldots \tag{2}$$

named after Edouard Lucas, a French mathematician of the 19th century, known also as the inventor of the Tower of Hanoi puzzle. Coincidentally, it was Lucas [12] who gave sequence (1) its name [9]. One can define Fibonacci numbers F_n and Lucas numbers L_n through the following definitions (difference equations)

$$\begin{aligned} F_{n+1} &= F_n + F_{n-1} \\ F_0 &= 1 \\ F_1 &= 1 \end{aligned} \tag{3}$$

and

$$\begin{aligned} L_{n+1} &= L_n + L_{n-1} \\ L_0 &= 2 \\ L_1 &= 1 \end{aligned} \tag{4}$$

The recursive nature of Fibonacci and Lucas numbers is often used as the demonstration of the spreadsheet facility to generate numbers through the so-called *ostensive* definition: given the first two terms, definition (3) (or (4)) is communicated to the spreadsheet by pointing at the givens. The spreadsheet that generates Fibonacci numbers is shown in Figure 1 where the formula bar includes definition (3) formulated in the language of cell references. By attaching sliders to the cells containing the first two terms, different Fibonacci-like sequences can be generated (e.g., Lucas numbers in Figure 2). One can say that Fibonacci-like sequences represent the most natural generalization of the classic Fibonacci number sequence (1). In terms of a spreadsheet, proceeding from Fibonacci numbers, one can learn using the tool for numerical modeling of difference equations, and in doing so to motivate the development of abstract mathematical concepts employing the power of numerical examples.

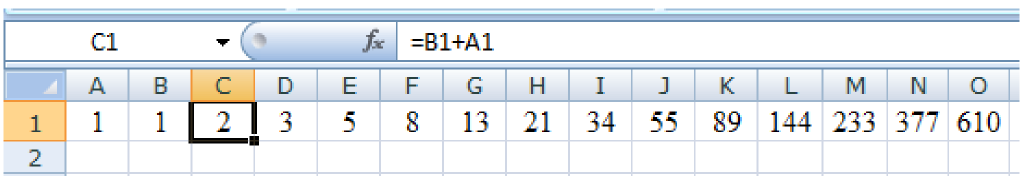


Figure 1: Fibonacci numbers.

3 Parameterization of Fibonacci recursion

The next idea to extend the context of Fibonacci numbers is to change the rule according to which the numbers develop. For example, consider the sequence

$$1, 1, 3, 7, 17, 41, \ldots \tag{5}$$

in which every number beginning from the third is twice the previous number plus the one that precedes it. Just like number sequences (1) and (2) can be formally defined

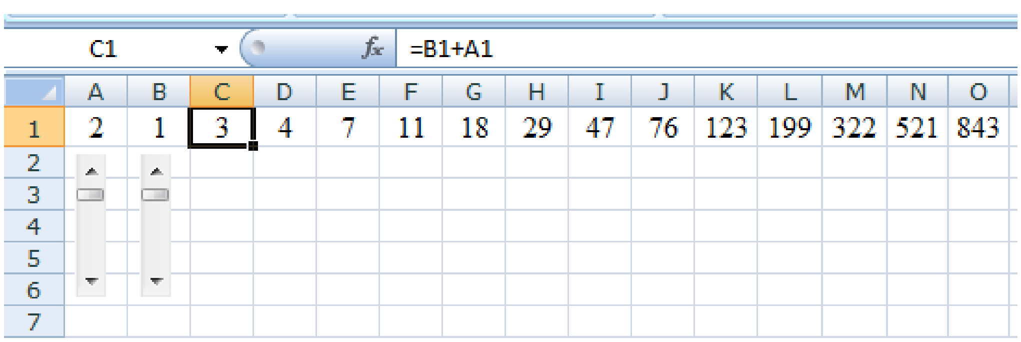


Figure 2: Lucas numbers.

through difference equations (3) and (4), sequence (5) satisfies the difference equation

$$\begin{aligned} f_{n+1} &= 2f_n + f_{n-1} \\ f_0 &= 1 \\ f_1 &= 1 \end{aligned} \tag{6}$$

The coefficient in f_n can be considered a parameter, and in the spreadsheet environment a slider-controlled cell can be given the name a so that sequence (5) and other number sequences (this time, we do not call them Fibonacci-like numbers) can be generated through the formula shown in the formula bar (Figure 3). The parameterization of

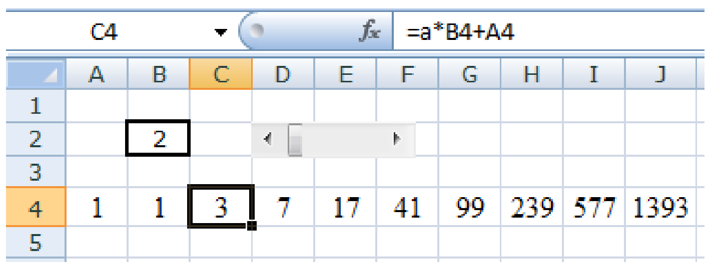


Figure 3: Modeling equation (6).

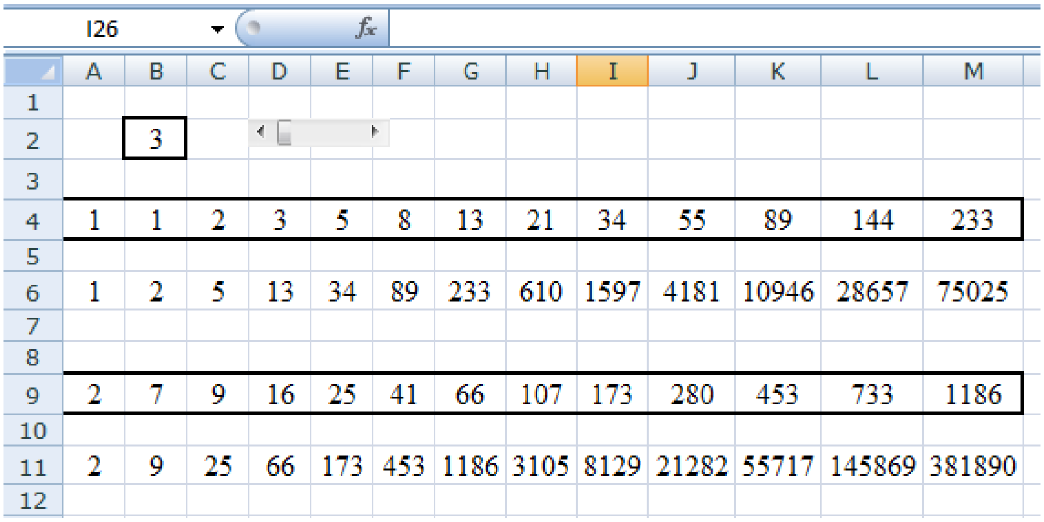
difference equations points to another avenue of extending the exploration of Fibonacci numbers through the use of a spreadsheet. Consider the sequence

$$1, 2, 5, 13, 34, 89, \ldots \tag{7}$$

obtained by eliminating every second number in sequence (1) Once can check to see that sequence (7) satisfies the difference equation

$$\begin{aligned} f_{n+1} &= 3f_n - f_{n-1} \\ f_0 &= 1 \\ f_1 &= 2 \end{aligned} \tag{8}$$

Note that, unlike sequence (5), which was introduced through a recursive definition (difference equation), sequence (7) resulted from some action on Fibonacci numbers so that the corresponding difference equation was motivated by this action. Put another way, a physical phenomenon is primary and its mathematical description (model) is secondary. By using the spreadsheet shown in Figure 4, one can discover that, regardless of initial values, eliminating every second number from any Fibonacci-like sequence yields the sequence of numbers described by difference equation (8). This is a quite unexpected phenomenon: by taking the first and the third terms of any Fibonacci-like sequence as the initial values for a new sequence developed through formula (8), one eliminates every second term of the original sequence. Below, equation (8) will be referred to as the Fibonacci sieve of order one. Now, one can introduce another generalization of Fibonacci


I26

	A	B	C	D	E	F	G	H	I	J	K	L	M
1													
2		3											
3													
4	1	1	2	3	5	8	13	21	34	55	89	144	233
5													
6	1	2	5	13	34	89	233	610	1597	4181	10946	28657	75025
7													
8													
9	2	7	9	16	25	41	66	107	173	280	453	733	1186
10													
11	2	9	25	66	173	453	1186	3105	8129	21282	55717	145869	381890
12													

Figure 4: Rows 6 and 11, respectively, generate every 2nd number of rows 4 and 9.

numbers by introducing parameters a, b, α, and β into difference equation (3) as follows

$$\begin{aligned} f_{n+1} &= af_n + bf_{n-1} \\ f_0 &= \alpha \\ f_1 &= \beta \end{aligned} \tag{9}$$

In the next section, it will be demonstrated how the introduction of such a generalization can be motivated by the development of the Fibonacci sieves of order greater than one. The spreadsheet that models equation (9) for different values of the parameters is shown in Figure 5 displaying the particular case of equation (8), i.e., the case $a = 3$, $b = 1$, $\alpha = 1, \beta = 2$. Note that in order to allow for both positive and negative integer values of parameters a and b, two kinds of sliders have to be used: a value slider and a sign slider. The sign slider is assigned two numerical values only: 0 as the minimum value and 1 as the maximum value. In the former case, the value slider (that can only be positive) is kept without change; in the latter case, the value slider is negated. As the variation of the initial values does not qualitatively chance the behavior of sequence (9), the values f_0 and f_1 can be manually controlled.

4 Fibonacci sieve generated by a linear difference equation

One may wonder as to what the values of parameters a and b are, if the process of the elimination of every second number, this time, from sequence (7), is repeated? To answer this question, one has to find a difference equation that generates the subsequence $1, 5, 34, 233, \ldots$ of Fibonacci numbers. The next exploration could be to eliminate every second number from the last sequence and find a model that describes the resulting number sequence. With this in mind, we will explore the general problem of finding

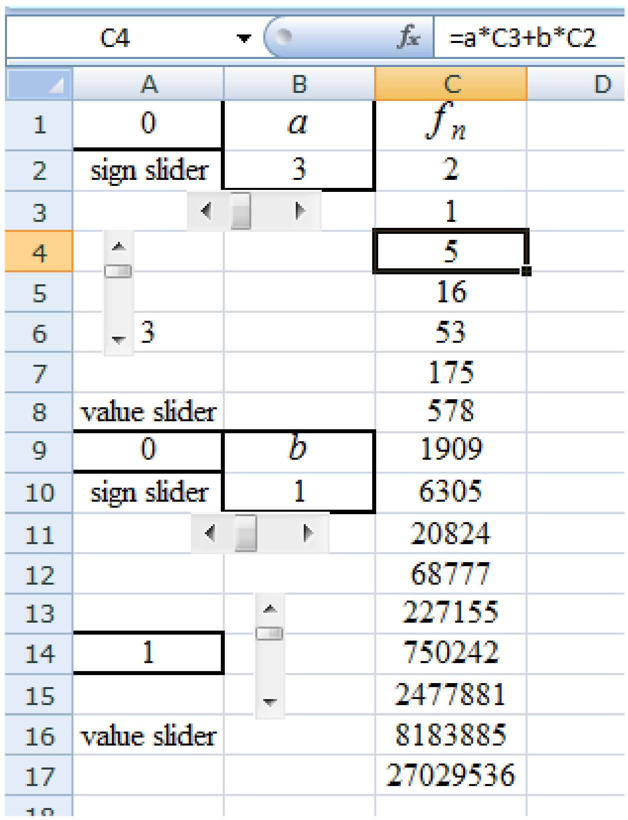


Figure 5: Modeling equation (9) for $a = 3$, $b = -1$, $\alpha = 1, \beta = 2$.

a difference equation that generates a subsequence F_{n_i} of Fibonacci numbers after the process of the elimination of every second number is applied to sequence (1) k times. Below, such a difference equation will be referred to as the Fibonacci sieve of order k. We begin with the sequence

$$F_0, F_1, F_2, \ldots, F_n, \ldots \tag{10}$$

and eliminate every second number from (10) to get a new sequence

$$F_0, F_2, F_4, \ldots, F_{2n}, \ldots \tag{11}$$

which, as was shown above, satisfies the equation

$$\begin{aligned} f_{n+1} &= 3f_n - f_{n-1} \\ f_0 &= F_0 \\ f_1 &= F_2 \end{aligned} \tag{12}$$

The second step is to eliminate every second number from (11) to get

$$F_0, F_4, F_8, \ldots, F_{4n}, \ldots \tag{13}$$

One can use the spreadsheet shown in Figure 6 to demonstrate (column `D`) that sequence (13) satisfies the equation

$$\begin{aligned} f_{n+1} &= 7f_n - f_{n-1} \\ f_0 &= F_0 \\ f_1 &= F_4 \end{aligned} \tag{14}$$

On the third step, eliminating every second number from sequence (13) yields the sequence

$$F_0, F_8, F_{16}, \ldots, F_{8n} \ldots \tag{15}$$

Once again, one can use the spreadsheet of Figure 6 to demonstrate (column `E`) that sequence (15) satisfies the equation

$$\begin{aligned} f_{n+1} &= 47f_n - f_{n-1} \\ f_0 &= F_0 \\ f_1 &= F_8 \end{aligned} \tag{16}$$

Note that the coefficients 3, 7, and 47 in recurrences (12), (14), and (16) are Lucas numbers L_2, L_4, and L_8, respectively. One can further conjecture that the next Lucas number in this sequence is L_{16}. This conjecture can be verified numerically by using the spreadsheet of Figure 6 (column `F`); cell `H5` displays Fibonacci number F_{64} – the smallest number that survives the Fibonacci sieve of order six. Alternatively, one can represent the Fibonacci sieve using the conditional formatting feature of the spreadsheet. The corresponding environment is shown in Figure 7. Here Fibonacci number F_{32} is the smallest number to survive the application of the Fibonacci sieve of order five.

n	$F_n \backslash f_{1,k}$	2	5	34	1597	3524578	17167680177565
0	1	1	1	1	1	1	1
1	1						
2	2	2					
3	3						
4	5	5	5				
5	8						
6	13	13					
7	21						
8	34	34	34	34			
9	55						
10	89	89					
11	144						
12	233	233	233				
13	377						
14	610	610					
15	987						
16	1597	1597	1597	1597	1597		
17	2584						
18	4181	4181					
19	6765						
20	10946	10946	10946				
21	17711						
22	28657	28657					
23	46368						
24	75025	75025	75025	75025			

Figure 6: The Fibonacci sieves of order one through six.

	A	B	C	D	E	F	G
1	**k=**	**0**	**1**	**2**	**3**	**4**	**5**
2	0	1	1	1	1	1	1
3	1	1					
4	2	2	2				
5	3	3					
6	4	5	5	5			
7	5	8					
8	6	13	13				
9	7	21					
10	8	34	34	34	34		
11	9	55					
12	10	89	89				
13	11	144					
14	12	233	233	233			
15	13	377					
16	14	610	610				
17	15	987					
18	16	1597	1597	1597	1597	1597	
19	17	2584					
20	18	4181	4181				
21	19	6765					
22	20	10946	10946	10946			
23	21	17711					
24	22	28657	28657				
25	23	46368					
26	24	75025	75025	75025	75025		
27	25	121393					
28	26	196418	196418				
29	27	317811					
30	28	514229	514229	514229			
31	29	832040					
32	30	1346269	1346269				
33	31	2178309					
34	32	3524578	3524578	3524578	3524578	3524578	3524578
35	33	5702887					
36	34	9227465	9227465				
37	35	14930352					
38	36	24157817	24157817	24157817			
39	37	39088169					
40	38	63245986	63245986				
41	39	102334155					
42	40	165580141	165580141	165580141	165580141		
43	41	267914296					
44	42	433494437	433494437				
45	43	701408733					
46	44	1134903170	1134903170	1134903170			
47	45	1836311903					
48	46	2971215073	2971215073				
49	47	4807526976					

Figure 7: Conditional formatting: F_{32} survived the Fibonacci sieve of order five.

Programming details of the two spreadsheet-based sieves are discussed in the Appendix. Another observation required for the development of the difference equation describing the Fibonacci sieve is that the numbers 2, 4, 8, and 16 (the ranks of the above four Lucas numbers) are the powers of two. Generalizing from the special cases yields

Proposition 1 *The family of difference equations*

$$\begin{aligned} f_{n+1,k} &= L_{2^k} f_{n,k} - f_{n-1,k} \\ f_{0,k} &= F_0 \\ f_{1,k} &= F_{2^k} \end{aligned} \tag{17}$$

where $k = 1, 2, 3, \ldots$, *and* $f_{n,k} = F_{n \cdot 2^k}$ *describe the Fibonacci sieve of order* k*. This proposition, motivated by spreadsheet modeling, will be proved in section 8 using Binet's formulas for Fibonacci and Lucas numbers.*

Equation (17) is a special case of equation (9) when $a = L_{2^k},\ b = -1,\ \alpha = F_0$, and $\beta = F_{2^k}$. The spreadsheet that generates solutions to equation (17) for different values of k is shown in Figure 8. It should be noted that any number generated by equation (17) is a Fibonacci number. Once again, one can use the spreadsheet pictured in Figure 5 to support the last statement. Because this is not the case for other integer values of a and b in equation (9), the family of difference equations (17) appears to be rather special. The last remark, in particular, raises the question if there exist other values of parameters a and b in equation (9) that yield different Fibonacci numbers only. The use of the word different in the last sentence is due to a trivial sequence $1, 1, 1, 1, \ldots$ that equation (9) generates for the initial values $\alpha = \beta = 1$ and any pair of parameters a and b satisfying the relation $a + b = 1$.

5 The Golden Ratio as an invariant for Fibonacci-like sequences

A spreadsheet can be further used to explore the behavior of the ratios F_{n+1}/F_n and L_{n+1}/L_n generated by difference equations (3) and (4), respectively (Figure 8, columns `G` and `H`). This simple exploration will demonstrate that regardless of the initial values of a Fibonacci-like sequence, as n grows larger, the ratios of two consecutive terms of the sequence approach the same number,

$$\frac{1+\sqrt{5}}{2} \ (= 1.6180334\ldots),$$

called the Golden Ratio. In much the same way, the behavior of the ratios f_{n+1}/f_n generated by equation (17) can be explored for different values of a and b. This exploration can be interpreted as yet another extension of the context of Fibonacci numbers leading to the notion of a generalized golden ratio. In order to explain the phenomenon

D1 =a*C1+b*B1

	A	B	C	D	E	F	G	H
1	f_n	1	34	1597	75025	4E+06	165580141	7.779E+09
2								
3	n	Fibonacci	Lucas	0	a		F_{n+1}/F_n	L_{n+1}/L_n
4	0	1	2	sign slider	47		1	0.5
5	1	1	1				2	3
6	2	2	3				1.5	1.3333333
7	3	3	4				1.6666667	1.75
8	4	5	7	47			1.6	1.5714286
9	5	8	11				1.625	1.6363636
10	6	13	18	value slider			1.6153846	1.6111111
11	7	21	29	1	b		1.6190476	1.6206897
12	8	34	47	sign slider	-1		1.6176471	1.6170213
13	9	55	76				1.6181818	1.6184211
14	10	89	123				1.6179775	1.6178862
15	11	144	199				1.6180556	1.6180905
16	12	233	322	1			1.6180258	1.6180124
17	13	377	521				1.6180371	1.6180422
18	14	610	843	value slider			1.6180328	1.6180308
19	15	987	1364				1.6180344	1.6180352
20	16	1597	2207				1.6180338	1.6180335
21	17	2584	3571	k	2^k		1.6180341	1.6180342
22	18	4181	5778	3	8		1.618034	1.6180339
23	19	6765	9349				1.618034	1.618034
24	20	10946	15127				1.618034	1.618034

Figure 8: The Fibonacci sieve of order three (row 1) and the Golden Ratio.

of convergence of the ratios to the Golden Ratio regardless of the initials values of a Fibonacci-like sequence, consider the sequence

$$x, y, x+y, x+2y, 2x+3y, 3x+5y, 5x+8y, 8x+13y, ... \tag{18}$$

which develops from the first two terms x and y through the rule that Fibonacci numbers follow. Setting $f_8(x,y)$ to be the 8th term of sequence (18), one can note that the coefficients in x and y are, respectively, the 6th and 7th Fibonacci numbers. Therefore $f_8(x,y) = F_6 \cdot x + F_7 \cdot y$. In general, using the method of mathematical induction, one can show that $f_n(x,y) = F_{n-2} \cdot x + F_{n-1} \cdot y$ is the n-th term of sequence (18). Noting that

$$\frac{F_n}{F_{n-1}} \xrightarrow[n\to\infty]{} \frac{1+\sqrt{5}}{2}$$

the ratios of two consecutive terms of sequence (18) tends to the Golden Ratio as well. Indeed,

$$\frac{f_{n+1}(x,y)}{f_n(x,y)} = \frac{F_{n-1} \cdot x + F_n \cdot y}{F_{n-2} \cdot x + F_{n-1} \cdot y} = \frac{F_{n-1} \cdot x(1 + \frac{F_n}{F_{n-1}} \cdot \frac{y}{x})}{F_{n-1} \cdot x(\frac{F_{n-2}}{F_{n-1}} + \frac{y}{x})} = \frac{1 + \frac{F_n}{F_{n-1}} \cdot \frac{y}{x}}{\frac{y}{x} + \frac{1}{\frac{F_{n-1}}{F_{n-2}}}}$$

$$\xrightarrow[n\to\infty]{} \frac{1 + \frac{1+\sqrt{5}}{2} \cdot \frac{y}{x}}{\frac{y}{x} + \frac{2}{1+\sqrt{5}}} = \frac{1+\sqrt{5}}{2} \cdot \frac{\frac{2}{1+\sqrt{5}} + \frac{y}{x}}{\frac{y}{x} + \frac{2}{1+\sqrt{5}}} = \frac{1+\sqrt{5}}{2}.$$

This explains that not only the ratios of two consecutive Lucas numbers converge to the Golden Ratio, but also the ratios of two consecutive terms of any Fibonacci-like sequence converge to the Golden Ratio. Put another way, the Golden Ratio turns out to be an invariant for the whole family of Fibonacci-like sequences.

6 The spreadsheet as an agency of new discoveries

In the case of equation (17), another interesting phenomenon can be observed through the use of a spreadsheet. Let $f_{n,k}$ represent the subsequence of Fibonacci numbers resulted from the application of the Fibonacci sieve of order k to sequence (1). Then

$$\lim_{n\to\infty} \frac{f_{n+1,k}}{f_{n,k}} = L_{2^k}$$

In other words, the sequence of the ratios of two consecutive iterations generated by equation (17) for different values of k tends to the Lucas number L_{2^k} as n increases. In that way, Lucas numbers which subscripts are equal to the powers of two can be interpreted as generalized golden ratios. Several other generalizations of the Golden Ratio are discussed in [16], [17]. Using a spreadsheet, one can also come across another unexpected phenomenon. When $a = 2$ and $b = -4$ in equation (9), the ratios f_{n+1}/f_n assume three values only: $1, -2$, and -4. In other words, generalized golden ratios can form cycles. In the last case, a cycle of period three was observed. How can these phenomena be explained? To answer this question, we need to solve difference equation (9) analytically, that is, to move from the recursive representation of the sequence f_n to its closed representation. In that way, whereas a spreadsheet becomes an agent of new discoveries, the tool, in turn, motivates formal mathematical activities described in the next section.

7 Developing a closed representation of the sequence f_n

We begin with replacing difference equation (9) by its matrix form

$$x_n = Ax_{n-1}, \quad x_0 = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \tag{19}$$

where

$$x_n = \begin{pmatrix} f_n \\ f_{n+1} \end{pmatrix}$$
$$A = \begin{pmatrix} 0 & 1 \\ b & a \end{pmatrix}$$

Iterating the right-hand side of matrix equation (19 yields

$$x_n = A^n x_0 \tag{20}$$

In order to find matrix A^n, one can substitute

$$A = S^{-1}BS \tag{21}$$

where B is the Jordanian form of matrix A. In that case,

$$\begin{aligned} x_n &= A^n x_0 \\ &= \underbrace{S^{-1}BSS^{-1}BSS^{-1}BS...S^{-1}BS}_{n} x_0 \\ &= S^{-1}B^n S x_0 \end{aligned}$$

In order to find matrix B, one has to find the eigenvalues of matrix A. To this end, one has to solve the characteristic equation $p^2 - ap - b = 0$ whose roots are

$$p_1 = \frac{a + \sqrt{a^2 + 4b}}{2}$$
$$p_2 = \frac{a - \sqrt{a^2 + 4b}}{2}$$

Therefore,

$$B = \begin{pmatrix} p_1 & 0 \\ 0 & p_2 \end{pmatrix}$$

and

$$B^n = \begin{pmatrix} {p_1}^n & 0 \\ 0 & {p_2}^n \end{pmatrix}$$

In order to find matrix

$$S = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix}$$

equation (21) can be written in the form

$$\begin{pmatrix} p_1 & 0 \\ 0 & p_2 \end{pmatrix} \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ b & a \end{pmatrix}$$

The latter can be reduced to the following system of four linear equations in four variables

$$\begin{aligned} S_{11} &= \frac{b}{p_1} = -p_2 \\ S_{12} &= 1 \\ S_{21} &= \frac{b}{p_2} = -p_1 \\ S_{22} &= 1 \end{aligned}$$

In that way,

$$S = \begin{pmatrix} -p_2 & 1 \\ -p_1 & 1 \end{pmatrix}$$

and

$$S^{-1} = \frac{1}{\sqrt{a^2+4b}} \begin{pmatrix} 1 & -1 \\ p_1 & -p_2 \end{pmatrix}$$

Now, we return to the solution space in order to find vector x_n.

$$\begin{aligned} x_n &= S^{-1} B^n S x_0 \\ &= \frac{1}{\sqrt{a^2+4b}} \begin{pmatrix} 1 & -1 \\ p_1 & -p_2 \end{pmatrix} \begin{pmatrix} p_1^n & 0 \\ 0 & p_2^n \end{pmatrix} \begin{pmatrix} -p_2 & 1 \\ -p_1 & 1 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \\ &= \frac{1}{\sqrt{a^2+4b}} \begin{pmatrix} 1 & -1 \\ p_1 & -p_2 \end{pmatrix} \begin{pmatrix} p_1^n & 0 \\ 0 & p_2^n \end{pmatrix} \begin{pmatrix} \beta - \alpha p_1 \\ \beta - \alpha p_2 \end{pmatrix} \\ &= \frac{1}{\sqrt{a^2+4b}} \begin{pmatrix} 1 & -1 \\ p_1 & -p_2 \end{pmatrix} \begin{pmatrix} -\alpha p_1^n p_2 + \beta p_1^n \\ -\alpha p_1 p_2^n + \beta p_2^n \end{pmatrix} \\ &= \frac{1}{\sqrt{a^2+4b}} \begin{pmatrix} -\alpha p_1^n p_2 + \beta p_1^n + \alpha p_1 p_2^n - \beta p_2^n \\ -\alpha p_1^{n+1} p_2 + \beta p_1^{n+1} + \alpha p_1 p_2^{n+1} - \beta p_2^{n+1} \end{pmatrix} \\ &= \frac{1}{\sqrt{a^2+4b}} \begin{pmatrix} p_1^n (\beta - \alpha p_2) - p_2^n (\beta - \alpha p_1) \\ p_1^{n+1} (\beta - \alpha p_2) - p_2^{n+1} (\beta - \alpha p_1) \end{pmatrix} \end{aligned}$$

Finally,

$$x_n = \begin{pmatrix} f_n \\ f_{n+1} \end{pmatrix} = \frac{1}{\sqrt{a^2+4b}} \begin{pmatrix} {p_1}^n(\beta - \alpha p_2) - {p_2}^n(\beta - \alpha p_1) \\ {p_1}^{n+1}(\beta - \alpha p_2) - {p_2}^{n+1}(\beta - \alpha p_1) \end{pmatrix}$$

whence

$$f_n = \frac{1}{\sqrt{a^2+4b}}({p_1}^n(\beta - \alpha p_2) - {p_2}^n(\beta - \alpha p_1)) \tag{22}$$

8 Binet's formulas and proof of Proposition 1

Formula (22) generates real numbers f_n when $a^2 + 4b \geq 0$. In the case of Fibonacci numbers F_n, substituting $a = b = 1$ and $\alpha = \beta = 1$ in formula (22) yields Binet's formula

$$\sqrt{5}F_n = \left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1} \tag{23}$$

In the case of Lucas numbers L_n, substituting $a = b = 1$, $\alpha = 2$, and $\beta = 1$ in formula (22) yields another Binet's formula

$$L_n = \left(\frac{1+\sqrt{5}}{2}\right)^{n} + \left(\frac{1-\sqrt{5}}{2}\right)^{n} \tag{24}$$

Note that formulas (23) and (24) as well as their development (as shown in the previous section) demonstrate a direct analogy with the integration of linear differential equations by means of reducing the corresponding system to the Jordanian form with the follow-up retreat to the original phase space. A pedagogical importance of making this observation in the context of the capstone course for secondary mathematics teachers is in enabling conceptual connections between the advanced mathematics the teachers are learning as undergraduates in the mathematics department and school mathematics they will be teaching [5]. Now one can prove Proposition 1 using, as above, the notation

$$p_1 = \frac{1+\sqrt{5}}{2}$$
$$p_2 = \frac{1-\sqrt{5}}{2}$$

According to formula (24),

$$L_{2^k} = {p_1}^{2^k} + {p_2}^{2^k} \tag{25}$$

According to formula (25),

$$f_{n,k} = F_{n2^k} = \frac{1}{\sqrt{5}}({p_1}^{n2^k+1} - {p_2}^{n2^k+1}) \tag{26}$$

$$f_{n-1,k} = F_{(n-1)2^k} = \frac{1}{\sqrt{5}}({p_1}^{(n-1)\cdot 2^k+1} - {p_2}^{(n-1)2^k+1}) \tag{27}$$

$$f_{n+1,k} = F_{(n+1)2^k} = \frac{1}{\sqrt{5}}({p_1}^{(n+1)2^k+1} - {p_2}^{(n+1)2^k+1}) \tag{28}$$

Substituting (25) – (27) in the right-hand side of difference equation (17) and noting $p_1p_2 = 1$, yields (28). Indeed,

$$\begin{aligned}
\sqrt{5}\left(L_{2^k} f_{n,k} - f_{n-1,k}\right) &= \left(p_1^{2^k} + p_2^{2^k}\right)\left(p_1^{n2^k+1} - p_2^{n2^k+1}\right) - \left(p_1^{(n-1)2^k+1} - p_2^{(n-1)2^k+1}\right) \\
&= p_1^{(n+1)2^k+1} - p_2^{(n+1)2^k+1} + p_1^{n2^k+1} p_2^{2^k} - p_1^{2^k} p_2^{n2^k+1} \\
&- \left(p_1^{(n-1)2^k+1} - p_2^{(n-1)2^k+1}\right) \\
&= \sqrt{5}F_{(n+1)2^k} + (p_1p_2)^{2^k}\left(p_1^{(n-1)2^k+1} - p_2^{(n-1)2^k+1}\right) \\
&- \left(p_1^{(n-1)2^k+1} - p_2^{(n-1)2^k+1}\right) \\
&= \sqrt{5}F_{(n+1)2^k} \\
&= \sqrt{5}f_{n+1,k}
\end{aligned}$$

This completes the proof of Proposition 1.

9 Limiting behavior of the ratios f_{n+1}/f_n

Note that if $a > 0$, then $|p_2| < |p_1|$ whence

$$\lim_{n\to\infty}\left(\frac{p_2}{p_1}\right)^n = 0$$

If $a < 0$, then $|p_1| < |p_2|$ whence

$$\lim_{n\to\infty}\left(\frac{p_1}{p_2}\right)^n = 0$$

Setting $C_1 = \beta - \alpha p_2$ and $C_2 = \alpha p_1 - \beta$ in formula (22), and assuming $C_1 \neq 0$ and $C_2 \neq 0$ yields

$$\frac{f_{n+1}}{f_n} = \frac{C_1p_1^{n+1} + C_2p_2^{n+1}}{C_1p_1^n + C_2p_2^n} = \frac{C_1p_1 + C_2p_2(\frac{p_2}{p_1})^n}{C_1 + C_2(\frac{p_2}{p_1})^n} = \frac{C_1p_1(\frac{p_1}{p_2})^n + C_2p_2}{C_1(\frac{p_1}{p_2})^n + C_2}$$

Therefore, if $a > 0$:

$$\lim_{n\to\infty}\frac{f_{n+1}}{f_n} = \lim_{n\to\infty}\frac{C_1p_1 + C_2p_2(\frac{p_2}{p_1})^n}{C_1 + C_2(\frac{p_2}{p_1})^n} = p_1 = \frac{a + \sqrt{a^2+4b}}{2}$$

and if $a < 0$:

$$\lim_{n\to\infty}\frac{f_{n+1}}{f_n} = \lim_{n\to\infty}\frac{C_1p_1(\frac{p_1}{p_2})^n + C_2p_2}{C_1(\frac{p_1}{p_2})^n + C_2} = p_2 = \frac{a - \sqrt{a^2+4b}}{2}$$

As was mentioned above, the case $a+b=1$ is a special one in a sense that $f_n = 1$ for all $n = 1, 2, 3, \ldots$. So, the above statements about the limiting values of the ratios f_{n+1}/f_n are true under the assumptions $a + b \neq 1$ and $a^2 + 4b \geq 0$.

10 Modeling the behavior of $f_{n+1,k}/f_{n,k}$ for different values of k

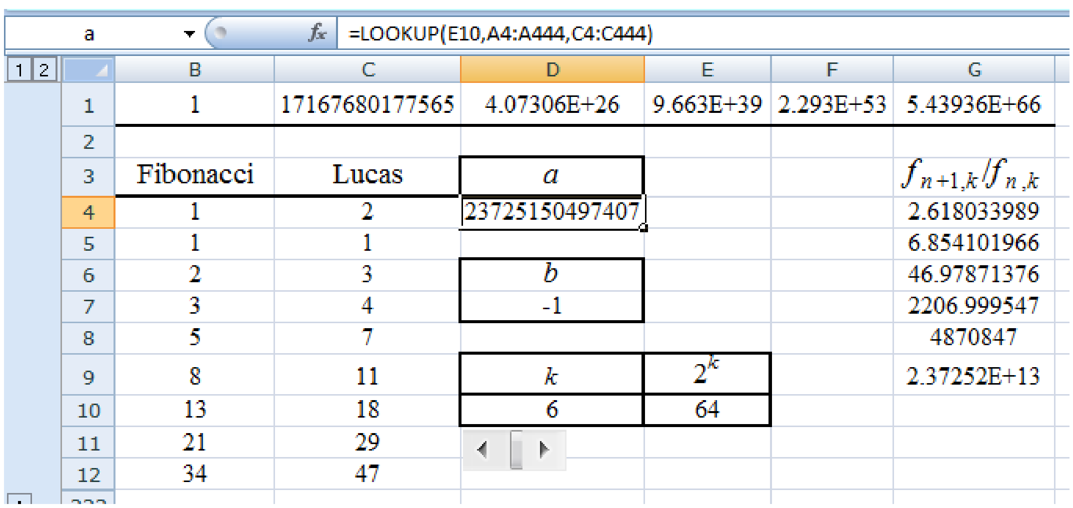
Now, for equation (17), one can refine the earlier conjecture about the limiting behavior of the ratios $f_{n+1,k}/f_{n,k}$. Indeed, applying the results of the previous section in the case $a = L_{2^k}$ and $b = -1$ yields the relationship

$$\lim_{n\to\infty} \frac{f_{n+1,k}}{f_{n,k}} = \frac{L_{2^k} + \sqrt{L_{2^k}^2 - 4}}{2}$$

The spreadsheet pictured in Figure 9 shows that, unlike the case of the Golden Ratio, the limiting values of the ratios $f_{n+1,k}/f_{n,k}$ (range `G4:G19`) increase as the order of the Fibonacci sieve increases. Furthermore, already

$$\left|\lim_{n\to\infty}(f_{n+1,3}/f_{n,3}) - L_{2^3}\right| < 0.022$$

Thus, in practice, the generalized golden ratio for the Fibonacci sieve of order $k \geq 3$ is indistinguishable from L_{2^k}.

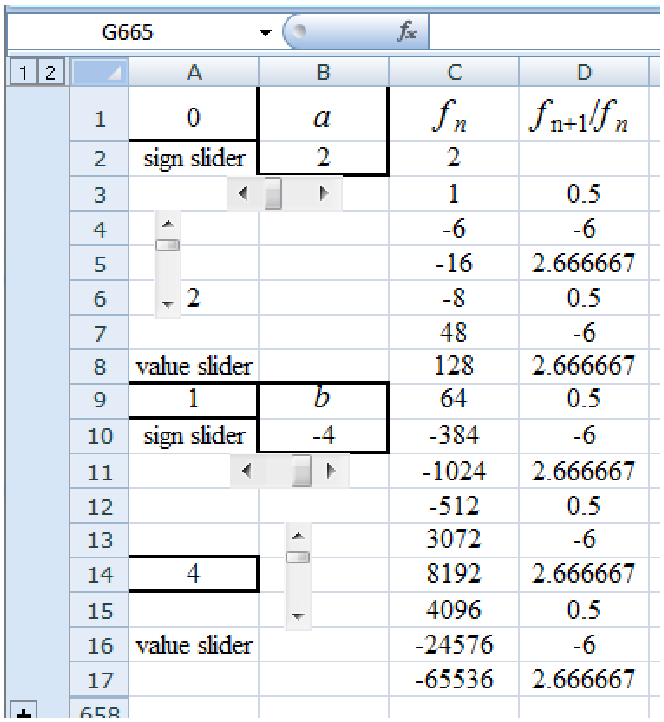

a | =LOOKUP(E10,A4:A444,C4:C444)

	B	C	D	E	F	G
1	1	17167680177565	4.07306E+26	9.663E+39	2.293E+53	5.43936E+66
2						
3	Fibonacci	Lucas	a			$f_{n+1,k}/f_{n,k}$
4	1	2	23725150497407			2.618033989
5	1	1				6.854101966
6	2	3	b			46.97871376
7	3	4	-1			2206.999547
8	5	7				4870847
9	8	11	k	2^k		2.37252E+13
10	13	18	6	64		
11	21	29				
12	34	47				

Figure 9: Modeling $\lim_{n\to\infty} \frac{f_{n+1,k}}{f_{n,k}}$ for $1 \leq k \leq 6$.

11 Exploring the case $a^2 + 4b < 0$

If $a^2 + 4b < 0$, the eigenvalues of matrix A are complex numbers and the right-hand side of formula (22) is undefined. Yet, regardless of the (real) values of a and b, equation (9) generates real numbers only. When $(a, b) = (2, -4)$ one has $a^2 + 4b < 0$, and, as was already mentioned above, the ratios f_{n+1}/f_n exhibit the cyclic behavior of period three. How can one find other values of a and b that provide the same type of the behavior of the ratios? Do there exist cycles of higher periods formed by the ratios? These questions point at another possible generalization of the notion of the Golden Ratio—a possibility for the cyclic behavior of generalized golden ratios. One can use the spreadsheet shown in Figure 10 to find other pairs of a and b for which the ratios form three-cycles. These are $(a, b) = (-3, -9), (-2, -4), (3, -9)$. One can further guess that these pairs of a and b satisfy the relation $a^2 + b = 0$, and confirm this guess through the use of the spreadsheet shown in Figure 10. In what follows, a mathematical model will be developed that allows for the rigorous determination of the relations between a and b for which the cyclic behavior of the ratios $f_{n+1}/f_n$of any given period occurs.



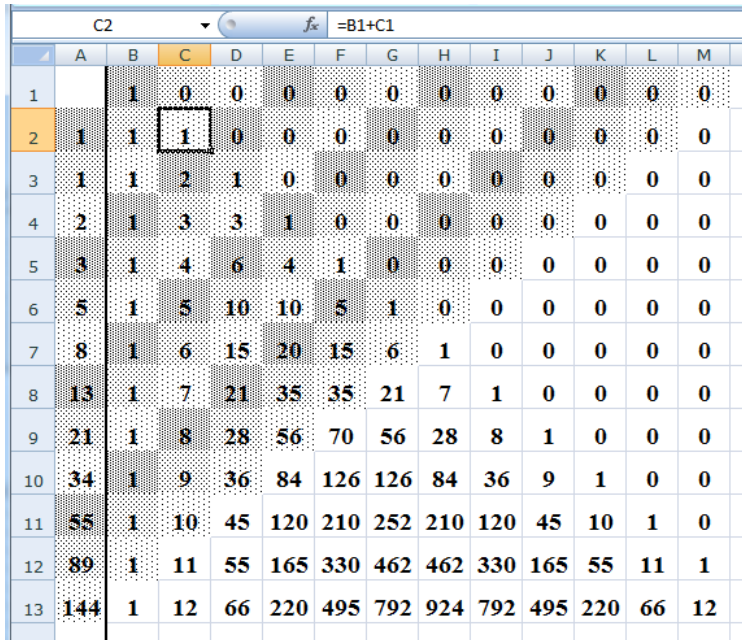
G665

	A	B	C	D
1	0	a	f_n	f_{n+1}/f_n
2	sign slider	2	2	
3			1	0.5
4			-6	-6
5			-16	2.666667
6	2		-8	0.5
7			48	-6
8	value slider		128	2.666667
9	1	b	64	0.5
10	sign slider	-4	-384	-6
11			-1024	2.666667
12			-512	0.5
13			3072	-6
14	4		8192	2.666667
15			4096	0.5
16	value slider		-24576	-6
17			-65536	2.666667

Figure 10: A generalized golden ratio forms a cycle of period three.

12 Fibonacci numbers and Pascal's triangle

Just like Fibonacci numbers, Pascal's triangle is another celebrated entity of discrete mathematics. It represents a triangular array of integers the n-th line of which is the sequence of the coefficients of $x^i y^{n-i}$ in the expansion of the binomial $(x + y)^n$, i.e., the sequence $\{C_n^{n-i}\}_{i=0}^n$. Using a spreadsheet, Pascal's triangle is more convenient to represent as a rectangular array (Figure 11). It is well-known that Fibonacci numbers can be found within Pascal's triangle. To this end, the binomial coefficients C_{n-i}^i have to be added along the so-called shallow diagonals of Pascal's triangle (Figure 11) with the



C2 =B1+C1

	A	B	C	D	E	F	G	H	I	J	K	L	M
1		1	0	0	0	0	0	0	0	0	0	0	0
2	1	1	1	0	0	0	0	0	0	0	0	0	0
3	1	1	2	1	0	0	0	0	0	0	0	0	0
4	2	1	3	3	1	0	0	0	0	0	0	0	0
5	3	1	4	6	4	1	0	0	0	0	0	0	0
6	5	1	5	10	10	5	1	0	0	0	0	0	0
7	8	1	6	15	20	15	6	1	0	0	0	0	0
8	13	1	7	21	35	35	21	7	1	0	0	0	0
9	21	1	8	28	56	70	56	28	8	1	0	0	0
10	34	1	9	36	84	126	126	84	36	9	1	0	0
11	55	1	10	45	120	210	252	210	120	45	10	1	0
12	89	1	11	55	165	330	462	462	330	165	55	11	1
13	144	1	12	66	220	495	792	924	792	495	220	66	12

Figure 11: Rectangular form of Pascal's triangle and shallow diagonals.

end points C_n^0 and C_{n-r}^r where, given $n, r = n/2$ or $r = (n+1)/2$. One can rearrange the elements of Pascal's triangle (in other words, to straighten out the shallow diagonals) so that the binomial coefficients C_{n-i}^i are positioned horizontally as shown in Figure 12. Then, the sum of the numbers in the n-th row of the rearranged Pascal's triangle is Fibonacci number F_{n+1} (the far-right column of the spreadsheet). One can also see that the sums with the same number of summands appear in pairs. In the first sum, the last summand is equal to one; in the second sum, the last summand is equal to the number of summands.

This computational approach to the discovery of Fibonacci numbers within Pascal's triangle can be used to motivate its formal demonstration.

Proposition 2

$$\sum_{i=0}^{n} C_{2n-i}^i = F_{2n}$$

and

$$\sum_{i=0}^{n} C_{2n-i+1}^i = F_{2n+1}$$

Proof. The case $n = 1$ yields

$$\sum_{i=0}^{1} C_{2-i}^i = 2$$

W8 f_x

	A	B	C	D	E	F	G	H	I	J	K	L	M	N
1	$n \setminus i$	0	1	2	3	4	5	6	7	8	9	10		
2	1	1	1											2
3	1	1	2											3
4	2	1	3	1										5
5	2	1	4	3										8
6	3	1	5	6	1									13
7	3	1	6	10	4									21
8	4	1	7	15	10	1								34
9	4	1	8	21	20	5								55
10	5	1	9	28	35	15	1							89
11	5	1	10	36	56	35	6							144
12	6	1	11	45	84	70	21	1						233
13	6	1	12	55	120	126	56	7						377
14	7	1	13	66	165	210	126	28	1					610
15	7	1	14	78	220	330	252	84	8					987
16	8	1	15	91	286	495	462	210	36	1				1597
17	8	1	16	105	364	715	792	462	120	9				2584
18	9	1	17	120	455	1001	1287	924	330	45	1			4181
19	9	1	18	136	560	1365	2002	1716	792	165	10			6765
20	10	1	19	153	680	1820	3003	3003	1716	495	55	1		10946
21	10	1	20	171	816	2380	4368	5005	3432	1287	220	11		17711

Figure 12: Rearrangement of the elements of Pascal's triangle.

and

$$\sum_{i=0}^{1} C_{3-i}^{i} = 3$$

Next, assuming that

$$\sum_{i=0}^{n} C_{2n-i}^{i} = F_{2n}$$

and

$$\sum_{i=0}^{n} C_{2n-i+1}^{i} = F_{2n+1}$$

and using the well-known combinatorial identity

$$C_m^k = C_{m-1}^k + C_{m-1}^{k-1} \tag{29}$$

(the spreadsheet-oriented geometric representation of which in pictured in Figure 14) one can show that

$$\sum_{i=0}^{n} C_{2n-i}^{i} + \sum_{i=0}^{n} C_{2n-i+1}^{i} = \sum_{i=0}^{n+1} C_{2(n+1)-i}^{i} = F_{2n+2}$$

Indeed,

$$\sum_{i=0}^{n} C_{2n-i}^{i} + \sum_{i=0}^{n} C_{2n-i+1}^{i} = (C_{2n}^0 + C_{2n}^1) + (C_{2n-1}^1 + C_{2n-1}^2) + ... + (C_{n+1}^{n-1} + C_{n+1}^n) + C_{2n+1}^0 + C_n^n$$
$$= C_{2n+1}^1 + C_{2n}^2 + ... + C_{n+2}^n + C_{2n+2}^0 + C_{n+1}^{n+1} = C_{2n+2}^0 + C_{2n+1}^1 + C_{2n}^2 + ... + C_{n+1}^{n+1}$$
$$= \sum_{i=0}^{n+1} C_{2n-i+2}^{i} = \sum_{i=0}^{n+1} C_{2(n+1)-i}^{i} = F_{2n+2}.$$

In other words, each term of the sequence F_k is the sum of the previous two terms and the first two terms are two and three. This completes the proof. ■

13 From Pascal's triangle to Fibonacci-like polynomials

The staircase-type arrangement of the binomial coefficients in the alternative representation of Pascal's triangle (Figure 13) prompts the idea to associate each step to which the coefficients belong with polynomials of the first, second, third, and so on degree, thereby, having two distinct polynomials of each degree. For example, the pair $\{C_2^0, C_1^1\}$ can be associated with the polynomial $P_1(x) = C_2^0 x + C_1^1$,the pair $\{C_3^0, C_2^1\}$ – with the polynomial $P_2(x) = C_3^0 x + C_2^1$, the triple $\{C_4^0, C_3^1, C_2^2\}$ – with the polynomial $P_3(x) = C_4^0 x^2 + C_3^1 x + C_2^2$, the triple $\{C_5^0, C_4^1, C_3^2\}$ – with the polynomial $P_4(x) = C_5^0 x^2 + C_4^1 x + C_3^2$, and so on. In such a way, the polynomials

$n\backslash i$	0	1	2	3	4	5
1	C_2^0	C_1^1				
1	C_3^0	C_2^1				
2	C_4^0	C_3^1	C_2^2			
2	C_5^0	C_4^1	C_3^2			
3	C_6^0	C_5^1	C_4^2	C_3^3		
3	C_7^0	C_6^1	C_5^2	C_4^3		
4	C_8^0	C_7^1	C_6^2	C_5^3	C_4^4	
4	C_9^0	C_8^1	C_7^2	C_6^3	C_5^4	
5	C_{10}^0	C_9^1	C_8^2	C_7^3	C_6^4	C_5^5
5	C_{11}^0	C_{10}^1	C_9^2	C_8^3	C_7^4	C_6^5

Figure 13: Binomial coefficients in the rearranged Pascal's triangle.

$$
\begin{aligned}
P_1(x) &= x + 1 \\
P_2(x) &= x + 2 \\
P_3(x) &= x^2 + 3x + 1 \\
P_4(x) &= x^2 + 4x + 3 \\
P_5(x) &= x^3 + 5x^2 + 6x + 1 \\
P_6(x) &= x^3 + 6x^2 + 10x + 4 \\
P_7(x) &= x^4 + 7x^3 + 15x^2 + 10x + 1 \\
P_8(x) &= x^4 + 8x^3 + 21x^2 + 20x + 5 \\
P_9(x) &= x^5 + 9x^4 + 28x^3 + 35x^2 + 15x + 1 \\
P_{10}(x) &= x^5 + 10x^4 + 36x^3 + 56x^2 + 35x + 6
\end{aligned}
$$

(to list all polynomials of degree one through five, two polynomials of each degree) can be developed. In general, one can introduce two types of polynomials

$$P_{2n-1}(x) = \sum_{i=0}^{n} C_{2n-i}^{i} x^{n-i} \tag{30}$$

and

$$P_{2n}(x) = \sum_{i=0}^{n} C_{2n-i+1}^{i} x^{n-i} \tag{31}$$

where $P_{2n-1}(0) = 1$ and $P_{2n}(0) = n + 1$. In addition, let us define $P_{-1}(x) = P_0(x) = 1$.

As will be shown below, polynomials (30) and (31) satisfy a single recursive formula

$$\begin{aligned} P_n(x) &= x^{n \bmod 2} P_{n-1}(x) + P_{n-2}(x) \\ P_0(x) &= 1 \\ P_1(x) &= x + 1 \end{aligned} \tag{32}$$

where $n \bmod 2$ is the remainder when n divided by 2. Following the terminology introduced elsewhere [1], polynomials (32) will be referred to below as Fibonacci-like polynomials.

Proposition 3 $P_n(1) = F_{n+1}$ *for all* $n = -1, 0, 1, 2, 3, \ldots$.

Proof. According to Proposition 2, $P_{2n-1}(1) = F_{2n}$ and $P_{2n}(1) = F_{2n+1}$. Substituting n for $2n$ completes the proof. ■

There are several classes of recursively-defined polynomials associated with Fibonacci numbers. For example, the polynomials

$$\begin{aligned} F_n(x) &= xF_{n-1}(x) + F_{n-2}(x) \\ F_0(x) &= 1 \\ F_1(x) &= x \end{aligned}$$

are referred to in [9] as Catalan's Fibonacci polynomials. Note that one property that polynomials $P_n(x)$ and $F_n(x)$ have in common is that both polynomials assume the values of Fibonacci numbers at $x = 1$. In many other aspects, the properties of the two classes of polynomials are quite different. For example, for each degree n there is only one Catalan's Fibonacci polynomial $F_n(x)$, whereas there are two Fibonacci-like polynomials $P_{2n-1}(x)$ and $P_{2n}(x)$ of the same degree n. Another class of polynomials associated with Fibonacci numbers that have more properties in common with Fibonacci-like polynomials was presented in 1918 to Berlin Mathematical Association by the German mathematician Jacobsthal. Referred to in [9] as Jacobsthal polynomials, they were defined [8] through the recurrence equation

$$\begin{aligned} J_{n+1}(x) &= J_n(x) + xJ_{n-1}(x), \\ J_0(x) &= 1 \\ J_1(x) &= 1 \end{aligned}$$

One can see that just like the polynomials $F_n(x)$ and $P_n(x)$, the polynomials $J_n(x)$ assume the values of Fibonacci numbers at $x = 1$. However, unlike the case of $F_n(x)$, there are also two polynomials $J_{2n-1}(x)$ and $J_{2n}(x)$ of the same degree n. Here are the first ten Jacobsthal polynomials:

$$J_0(x) = 1$$
$$J_1(x) = 1$$
$$J_2(x) = x + 1$$
$$J_3(x) = 2x + 1$$
$$J_4(x) = x^2 + 3x + 1$$
$$J_5(x) = 3x^2 + 4x + 1$$
$$J_6(x) = x^3 + 6x^2 + 5x + 1$$
$$J_7(x) = 4x^3 + 10x^2 + 6x + 1$$
$$J_8(x) = x^4 + 10x^3 + 15x^2 + 7x + 1$$
$$J_9(x) = 5x^4 + 20x^3 + 21x^2 + 8x + 1$$

Interestingly enough, a Jacobsthal polynomial of degree n can be transformed into the corresponding Fibonacci-like polynomial by swapping coefficients in the powers of x with the sum of exponents equal to n. In that way, as will be shown below, whereas the roots of Fibonacci-like polynomials are responsible for the cyclic behavior of generalized golden ratios, the roots of Jacobsthal polynomials do not provide such a phenomenon. Yet the reciprocals of the roots of a Jacobsthal polynomial are the roots of the corresponding Fibonacci-like polynomial. It is important to emphasize, however, that Fibonacci-like polynomials $P_n(x)$ were discovered by the authors in the applied context of exploring qualitative behavior of solutions of a two-parametric linear discrete system. That is, polynomials $P_n(x)$ were first found as the means of connecting classic and modern mathematical concepts. On the other hand, Jacobsthal polynomials were introduced as a generalization of Fibonacci numbers followed by the demonstration of connections between the polynomials and some number theoretical functions [8]. Several applications of Catalan's Fibonacci polynomials to graph theory can be found in [7].

14 Polynomial generalizations of the Fibonacci sieve of order one

One can recognize that coefficients in the same powers of x for different polynomials $Pn(x)$ are related. For example, the sum of the coefficients of x^3 of the polynomials $P_9(x)$ and $P_8(x)$ is equal to 36—the coefficient of x^3 of the polynomial $P_{10}(x)$. Similar relationships can be observed for other powers of x of three consecutive polynomials. This simple observation can motivate

Proposition 4 *For all* $n = 1, 2, 3, 4, \ldots$

$$P_{2n}(x) = P_{2n-1}(x) + P_{2n-2}(x) \qquad (33)$$

In other words, every Fibonacci-like polynomial with an even second coefficient is equal to the sum of the previous two polynomials (one of which is of the same degree and another polynomial of one degree lower with an even second coefficient). In terms of relation (32),

$$P_{2n}(x) = x^{n \bmod 2} P_{2n-1}(x) + P_{2n-2}(x) = P_{2n-1}(x) + P_{2n-2}(x)$$

Proof. To prove identity (33), it can be first re-written in the form

$$\sum_{i=0}^{n} C_{2n-i+1}^{i} x^{n-i} = \sum_{i=0}^{n} C_{2n-i}^{i} x^{n-i} + \sum_{i=0}^{n} C_{2n-i-1}^{i} x^{n-i-1}$$

Expanding the sums and applying identity (29) to the binomial coefficients in the left-hand side of the last relation yields

$$\begin{aligned} &C_{2n+1}^{0} x^{n} + (C_{2n-1}^{1} + C_{2n-1}^{0}) x^{n-1} + (C_{2n-2}^{2} + C_{2n-2}^{1}) x^{n-2} + ... + C_{n}^{n} + C_{n}^{n-1} \\ &= C_{2n}^{0} x^{n} + (C_{2n-1}^{1} + C_{2n-1}^{0}) x^{n-1} + (C_{2n-2}^{2} + C_{2n-2}^{1}) x^{n-2} + ... + C_{n}^{n} + C_{n}^{n-1} \end{aligned}$$

Noting that $C_{2n+1}^{0} = C_{2n}^{0}$ competes the proof of identity (33). ■

Remark 1 Identity (33) holds neither for Catalan's Fibonacci polynomials $F_n(x)$ nor for Jacobsthal polynomials $J_n(x)$.

Proposition 5 *For all* $n = 1, 2, 3, \ldots$

$$\sum_{i=0}^{n} P_{2i-1}(x) = P_{2n}(x) \tag{34}$$

Proof. Applying the method of mathematical induction and identity (33) yields formula (34). ■

Corollary 1 The identity

$$\sum_{i=0}^{n} F_{2i} = F_{2n+1}$$

for Fibonacci numbers F_n holds true.

Proof. Substituting $x = 1$ in formula (34) and using Proposition 3 completes the proof. ■

Proposition 6 *For all* $n = 1, 2, 3, \ldots$

$$P_{2n+1}(x) = xP_{2n}(x) + P_{2n-1}(x) \tag{35}$$

In terms of relation (32),

$$P_{2n+1}(x) = x^{(2n+1) \bmod 2} P_{2n}(x) + P_{2n-1}(x) = xP_{2n}(x) + P_{2n-1}(x)$$

Proof. In order to prove identity (35), one has to show that

$$xP_{2n}(x) + P_{2n-1}(x) = x\sum_{i=0}^{n} C_{2n-i+1}^{i}x^{n-i} + \sum_{i=0}^{n} C_{2n-i}^{i}x^{n-i}$$
$$= \sum_{i=0}^{n}(C_{2n-i+1}^{i}x^{n-i+1} + C_{2n-i}^{i}x^{n-i}) = \sum_{i=0}^{n+1} C_{2n-i+2}^{i}x^{n-i+1} = P_{2n+1}(x).$$

Indeed,

$$C_{2n+1}^{0}x^{n+1} + C_{2n}^{1}x^{n} + C_{2n-1}^{2}x^{n-1} + ... + C_{n}^{n}x$$
$$+C_{2n}^{0}x^{n} + C_{2n-1}^{1}x^{n-1} + C_{2n-2}^{2}x^{n-2} + ... + C_{n}^{n}$$
$$= C_{2n+1}^{0}x^{n+1} + (C_{2n}^{1} + C_{2n}^{0})x^{n} + (C_{2n-1}^{2} + C_{2n-1}^{1})x^{n-1} + ... + (C_{n+1}^{n} + C_{n+1}^{n-1})x + C_{n}^{n}$$
$$= C_{2n+1}^{0}x^{n+1} + C_{2n+1}^{1}x^{n} + C_{2n}^{2}x^{n-1} + ... + C_{n+2}^{n}x + C_{n}^{n}$$
$$= C_{2n+2}^{0}x^{n+1} + C_{2n+2-1}^{1}x^{n+1-1} + C_{2n+2-2}^{2}x^{n+1-2} + ... + C_{2n+2-n}^{n}x^{n+1-n} + C_{2n+2-(n+1)}^{n+1}$$
$$= \sum_{i=0}^{n+1} C_{2n-i+2}^{i}x^{n-i+1}.$$

This completes the proof of identity (35). ■

Remark 2 Identities (33) and (35) are in agreement with recursive definition (32) of Fibonacci-like polynomials $P_n(x)$.

Remark 3 Identity (35) holds for Catalan's Fibonacci polynomials $F_n(x)$ and does not hold for Jacobsthal polynomials $J_n(x)$.

Proposition 7 *For all* $n = 1, 2, 3, \ldots$

$$x\sum_{i=0}^{n} P_{2i}(x) = P_{2n+1}(x) - 1 \qquad (36)$$

Proof. Applying mathematical induction and identity (35) yields formula (36). ■

Corollary 2 The identity

$$\sum_{i=0}^{n} F_{2i+1} = F_{2n+2} - 1$$

for Fibonacci numbers F_n holds true.

Proof. Substituting $x = 1$ in formula (36) and using Proposition 3 completes the proof. ■

Proposition 8 *For all* $n = 2, 3, 4, \ldots$

$$P_{2n-1}(x) = P_{2n-3}(x)P_2(x) - P_{2n-5}(x) \qquad (37)$$

In other words, three consecutive Fibonacci-like polynomials with an odd second coefficient are related to one another by a recursive formula.

Proof. To prove recursive formula (37), it can be first re-written in the form

$$\sum_{i=0}^{n} C_{2n-i}^{i} x^{n-i} = (x+2)\sum_{i=0}^{n} C_{2n-2-i}^{i} x^{n-1-i} + \sum_{i=0}^{n} C_{2n-4-i-1}^{i} x^{n-2-i}$$

Expanding the sums yields

$$\begin{aligned}&C_{2n}^{0}x^{n} + C_{2n-1}^{1}x^{n-1} + C_{2n-2}^{2}x^{n-2} + ... + C_{n}^{n} \\ &= (x+2)(C_{2n-2}^{0}x^{n-1} + C_{2n-3}^{1}x^{n-2} + C_{2n-4}^{2}x^{n-3} + ... + C_{n-1}^{n-1}) \\ &-(C_{2n-4}^{0}x^{n-2} + C_{2n-5}^{1}x^{n-3} + ... + C_{n-2}^{n-2}).\end{aligned}$$

Next, arranging the right-hand side of the last relation as a polynomial in the powers of x, results in the following equality between two polynomials of degree n

$$\begin{aligned}&C_{2n}^{0}x^{n} + C_{2n-1}^{1}x^{n-1} + C_{2n-2}^{2}x^{n-2} + ... + C_{n}^{n} \\ &= C_{2n-2}^{0}x^{n} + (C_{2n-3}^{1} + 2C_{2n-2}^{0})x^{n-1} + (C_{2n-4}^{2} + 2C_{2n-3}^{1} - C_{2n-4}^{0})x^{n-2} \\ &+(C_{2n-5}^{3} + 2C_{2n-4}^{2} - C_{2n-5}^{1})x^{n-3} + ... + 2C_{n-1}^{n-1} - C_{n-2}^{n-2}\end{aligned}$$

Comparing coefficients in the corresponding powers of x in the left- and right-hand sides of the last equality, one has to prove that for all $i = 0, 1, \ldots, n$

$$C_{2n-i}^{i} = C_{2(n-1)-i}^{i} + 2C_{2n-1-i}^{i-1} - C_{2(n-1)-i}^{i-2} \tag{38}$$

As discussed in [4], many combinatorial identities can be proved by using the geometric (spreadsheet-oriented) representation of identity (29) in which three cells of a spreadsheet, representing the binomial coefficients C_m^k, C_{m-1}^k, and C_{m-1}^{k-1}, form the L-shape (Figure 14). Then each term of an identity to be proved, in our case identity (38), can be associated with the unique cell of the spreadsheet. The geometric proof involves a repeated application of the L-shape rule until cells that represent the identity in question become connected and all intermediate cells are eliminated. Applying this strategy to identity (38) and using notation shown in Figure 15 yields

$$A = B + C = B + D + E = B + D + B - F = D + 2B - F$$

where

$$\begin{aligned}A &= C_{2n-i}^{i} \\ D &= C_{2(n-1)-i}^{i} \\ B &= C_{2n-1-i}^{i-1} \\ F &= C_{2(n-1)-i}^{i-2}\end{aligned}$$

This completes the proof of Proposition 8. ■

Remark 4 Identity (37) holds neither for Catalan's Fibonacci polynomials $F_n(x)$ nor for Jacobsthal polynomials $J_n(x)$.

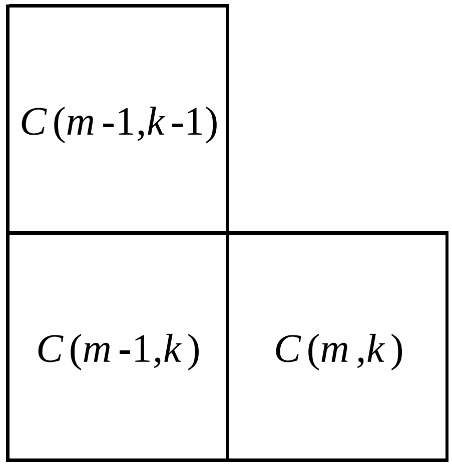


Figure 14: Geometric representation of identity (33).

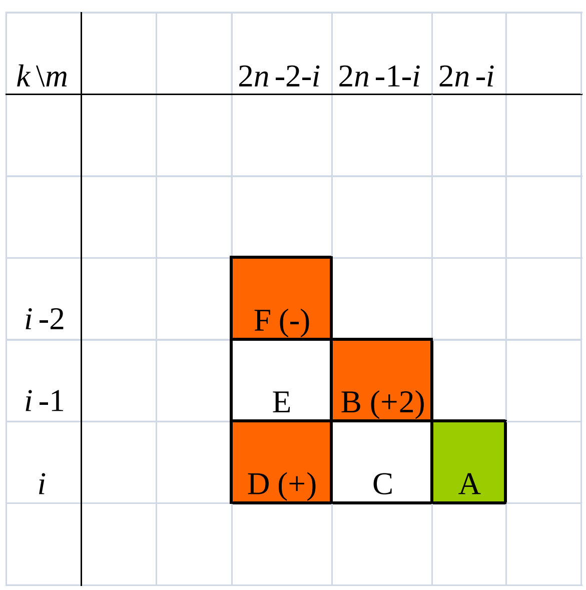


Figure 15: Geometric representation of identity (38).

Corollary 3 The identity

$$F_{2n} = 3F_{2n-2} - F_{2n-4} \tag{39}$$

holds true among three consecutive Fibonacci numbers with even subscripts.

Remark 5 Setting $f_n = F_{2n}$, one can recognize that identity (39) is equivalent to difference equation (12) which represents the Fibonacci sieve of order one. In that way, identity (37) for Fibonacci-like polynomials represents a polynomial generalization of the Fibonacci sieve of order one.

Proposition 9 *For all* $n = 2, 3, 4, \ldots$

$$P_{2n}(x) = P_{2n-2}(x)P_2(x) - P_{2n-4}(x) \tag{40}$$

In other words, three consecutive Fibonacci-like polynomials with an even second coefficient are also related to one another by a recursive formula.

Proof. Note that the notation $P_{2n}(x)$ was used to describe the sum in the left-hand side of equation (31). Expanding this sum yields

$$P_{2n}(x) = C^0_{2n+1}x^n + C^1_{2n}x^{n-1} + C^2_{2n-1}x^{n-2} + ... + C^n_{n+1} \tag{41}$$

One can use notation (41) to rewrite (40) as follows

$$\begin{aligned} &C^0_{2n+1}x^n + C^1_{2n}x^{n-1} + C^2_{2n-1}x^{n-2} + ... + C^n_{n+1} \\ &+C^0_{2n-3}x^{n-2} + C^1_{2n-4}x^{n-3} + C^2_{2n-5}x^{n-4} + ... + C^{n-2}_{n-1} \\ &= (C^0_{2n-1}x^{n-1} + C^1_{2n-2}x^{n-2} + C^2_{2n-3}x^{n-3} + ... + C^{n-1}_n)(x+2). \end{aligned} \tag{42}$$

Representing both sides of (42) as polynomials in the powers of x yields

$$\begin{aligned} &C^0_{2n+1}x^n + C^1_{2n}x^{n-1} + (C^2_{2n-1} + C^0_{2n-3})x^{n-2} + (C^3_{2n-2} + C^1_{2n-4})x^{n-3} + ... + (C^n_{n+1} + C^{n-2}_{n-1}) \\ &= C^0_{2n-1}x^n + C^1_{2n-2}x^{n-1} + C^2_{2n-3}x^{n-2} + ... + C^{n-1}_n x \\ &+2(C^0_{2n-1}x^{n-1} + C^1_{2n-2}x^{n-2} + C^2_{2n-3}x^{n-3} + ... + C^{n-1}_n), \end{aligned}$$

or

$$\begin{aligned} &C^0_{2n+1}x^n + C^1_{2n}x^{n-1} + (C^2_{2n-1} + C^0_{2n-3})x^{n-2} + (C^3_{2n-2} + C^1_{2n-4})x^{n-3} + ... + (C^n_{n+1} + C^{n-2}_{n-1}) \\ &= C^0_{2n}x^n + (C^1_{2n-1} + 2C^0_{2n})x^{n-1} + (C^2_{2n-2} + 2C^1_{2n-1})x^{n-2} + ... + 2C^{n-1}_n. \end{aligned}$$

Therefore, one has to prove that for all $i = 0, 1, 2, \ldots, n$, the following identity holds true

$$C^{i+1}_{2n-i} + C^{i-1}_{2n-i-2} = C^{i+1}_{2n-i-2} + 2C^i_{2n-i-1} \tag{43}$$

Setting

$$A = C^{i+1}_{2n-i},\ F = C^{i-1}_{2n-i-2},\ D = C^{i+1}_{2n-i-2},\ B = C^i_{2n-i-1}$$

identity (43) takes the form $A + F = D + 2B$. The last relation can be proved geometrically (Figure 16) by repeated application of the L-shape rule as follows $A + F = C + B + F = D + E + B + F = D + B - F + B - F = D + 2B$. The above geometric proof of identity (43) can be complemented by its formal proof. To this end, setting $2n - i = m$ in (43) yields

$$C_m^{i+1} - 2C_{m-1}^i = C_{m-2}^{i+1} - C_{m-2}^{i-1} \tag{44}$$

Applying identity (29) to relation (44) results in

$$C_{m-1}^{i+1} + C_{m-1}^i - 2C_{m-1}^i = C_{m-2}^{i+1} - C_{m-2}^{i-1}$$

whence

$$C_{m-1}^{i+1} - C_{m-1}^i = C_{m-2}^{i+1} - C_{m-2}^{i-1}$$

Once again, using (29), the last equality can be re-written in the form of a true identity

$$C_{m-2}^{i+1} + C_{m-2}^i - C_{m-2}^i - C_{m-2}^{i-1} = C_{m-2}^{i+1} - C_{m-2}^{i-1}$$

This completes the proof of identity (43), thereby, proving Proposition 9. ■

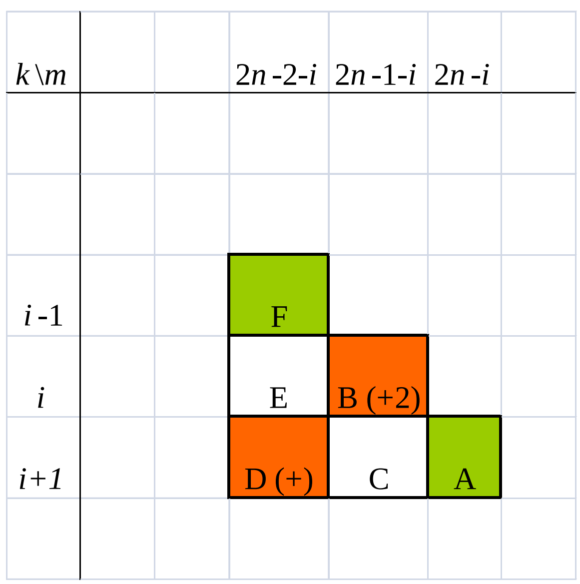


Figure 16: Geometric proof of identity (43) in a spreadsheet environment.

Remark 6 Identity (40) does not hold for Catalan's Fibonacci polynomials $F_n(x)$ as well as for Jacobsthal polynomials $J_n(x)$.

Corollary 4 The following identity

$$F_n = 3F_{n-2} - F_{n-4} \tag{45}$$

for Fibonacci numbers F_n $(n \geq 4)$ holds true.

Proof. Substituting $x = 1$ in formula (40) yields $F_{2n+1} = 3F_{2n-1} - F_{2n-3}$. This along with identity (39) completes the proof. ■

Remark 7 Identity (45) extends identity (39) to Fibonacci numbers with odd subscripts. In other words, replacing in (12) $f_0 = F_1$ and $f_1 = F_3$ yields the application of the Fibonacci sieve of order one to sequence (1), thereby, eliminating all Fibonacci numbers with even subscripts. In that way, identity (40) for Fibonacci-like polynomials represents a polynomial generalization of the Fibonacci sieve of order one with altered initial values.

15 Spreadsheet modeling of Fibonacci-like polynomials

Identity (29) can be used to generate coefficients of Fibonacci-like polynomials within a spreadsheet. Consider Figure 13 in which the triple $\{C_3^1,\ C_3^2,\ C_4^2\}$ of binomial coefficients satisfies identity (29). Setting $d(k,i)$ to represent a coefficient of a Fibonacci-like polynomial located in the cell of a spreadsheet with the coordinates (k,i), the coefficients of the polynomials can be generated through the spreadsheet modeling of the partial difference equation

$$d(k,i) = d(k-1,i) + d(k-2,i-1)$$

subject to the initial conditions

$$d(k,0) = 1,\ d(0,1) = 1,\ d(1,1) = 2,\ d(0,i) = d(1,i) = 1,\ i \geqslant 2$$

Then, the values of $P_n(x)$ can be tabulated in the interval $(-4,0)$. The chart pictured in Figure 18 shows the graphs of the first twenty Fibonacci-like polynomials $(1 \leq n \leq 10)$. One can recognize that some Fibonacci-like polynomials have common roots. Furthermore, it appears that the roots of Fibonacci-like polynomials have higher density near the endpoints of the interval $(-4,0)$, nonetheless being repelled by the endpoints.

16 Connecting Fibonacci-like polynomials to cycles

One of the most interesting properties of Fibonacci-like polynomials is their connection to the cyclic behavior of the ratios f_{n+1}/f_n generated by difference equation (9) in the case $a^2+4b < 0$. As was discovered earlier through spreadsheet modeling, the ratios form cycles of period three when $a^2 + b = 0$. Substituting $x = a^2/b$ in the last relation yields the equation $x+1 = 0$, that is, the equation $P_1(x) = 0$. Therefore, when $a^2/b = -1$, the ratios f_{n+1}/f_n generated by equation (9) form cycles of period three. In other words, for any $a \in \Re$ the pair $(a, -a^2)$ provides equation (9) with a three-cycle behavior of the ratios f_{n+1}/f_n. In much the same way, the equation $P_2(x) = 0$ can be connected to the relation $a^2+2b = 0$ which is responsible for the cycles of period four. Next, the equation $P_3(x) = 0$ can be connected to the relation $a^4+3a^2b+b^2 = 0$ which is responsible for the cycles of period five, and so on. In general, as shown in detail elsewhere [2], if $x = x_0$ is a zero of the polynomial $P_n(x)$, the relation $a^2/b = x_0$ determines those coordinates (a,b)

	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U
1		1	1	2	2	3	3	4	4	5	5	6	6	7	7	8	8	9	9	10	10
2		1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
3		1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
4		0	0	1	3	6	10	15	21	28	36	45	55	66	78	91	105	120	136	153	171
5		0	0	0	0	1	4	10	20	35	56	84	120	165	220	286	364	455	560	680	816
6		0	0	0	0	0	0	1	5	15	35	70	126	210	330	495	715	1001	1365	1820	2380
7		0	0	0	0	0	0	0	0	1	6	21	56	126	252	462	792	1287	2002	3003	4368
8		0	0	0	0	0	0	0	0	0	0	1	7	28	84	210	462	924	1716	3003	5005
9		0	0	0	0	0	0	0	0	0	0	0	0	1	8	36	120	330	792	1716	3432
10		0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	9	45	165	495	1287
11		0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	10	55	220
12		0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	11

Figure 17: Alternative modeling of Fibonacci-like polynomials.

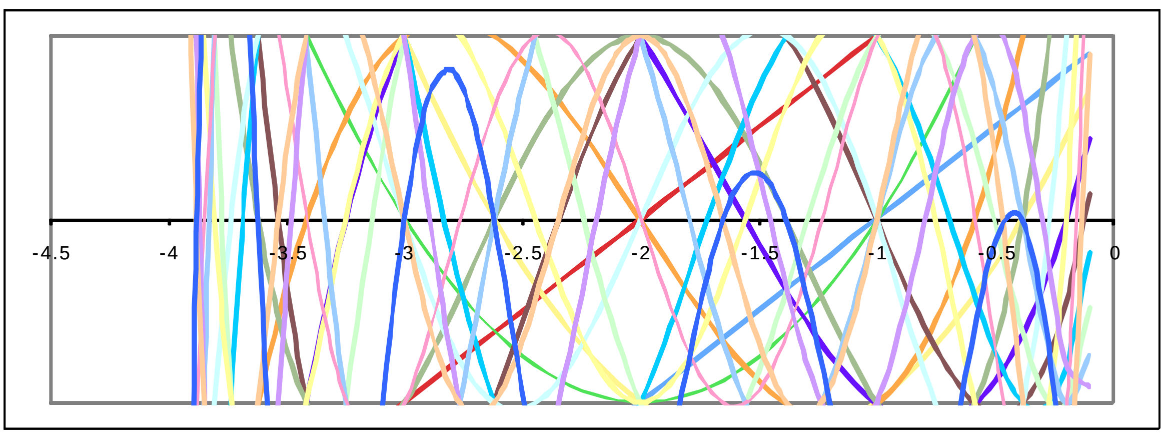


Figure 18: Graphs of Fibonacci-like polynomials.

in the plane of the parameters of equation (9) that provide cycles of period $n+2$ formed by the ratios f_{n+1}/f_n. However, as the degree of Fibonacci-like polynomials increases, the process of their root finding cannot be always carried out analytically, although, as was mentioned above, Fibonacci-like polynomials of higher degree may have roots in common with Fibonacci-like polynomials of the lower degree. For example, for all $n = 0, 1, 2, \ldots$ the polynomials $P_{3n+1}(x)$ and the polynomial $P_1(x) = x + 1$ share the root $x = -1$. In particular, the last statement implies that the generalized golden ratios can form cycles of arbitrary large period. In other cases, the spreadsheet-based method of iterations [3], [11] can be used a root finding algorithm. The application of this method to Fibonacci-like polynomials is discussed in the next section.

17 Finding the roots of $P_n(x)$ through the method of iterations

Proceeding from formula (30), the equation $P_{2n-1}(x) = 0$ can be re-written in the form

$$x^n + \sum_{i=1}^{n} C_{2n-i}^{i} x^{n-i} = 0$$

Dividing both sides of the last equation by x^{n-1} yields

$$x = -\sum_{i=1}^{n} \frac{C_{2n-i}^{i}}{x^{i-1}} \tag{46}$$

Likewise, using formula (31), the equation $P_{2n}(x) = 0$ can be transformed to the form

$$x = -\sum_{i=1}^{n} \frac{C_{2n-i+1}^{i}}{x^{i-1}} \tag{47}$$

According to the method of iterations, one has to choose the value $x = x_1$ as the initial approximation to a solution to equation (46) so that the value

$$x_2 = -\sum_{i=1}^{n} \frac{C_{2n-i}^{i}}{{x_1}^{i-1}}$$

becomes the second approximation to this solution. In much the same way, the value

$$x_2 = -\sum_{i=1}^{n} \frac{C_{2n-i+1}^{i}}{{x_1}^{i-1}}$$

becomes the second approximation to the solution of equation (47). In general, the dependence between two successive approximations to the solutions of equations (46) and (47) is given, respectively, by the formulas

$$x_{j+1} = -\sum_{i=1}^{n} \frac{C_{2n-i}^{i}}{{x_j}^{i-1}} \tag{48}$$

and

$$x_{j+1} = -\sum_{i=1}^{n} \frac{C_{2n-i+1}^{i}}{{x_j}^{i-1}} \tag{49}$$

Suppose that after j iterations the equality $x_{j+1} = x_j$ is satisfied with the specified precision. This means that x_j is an approximate value of the root of equation (46) (or (47)) with that precision. Whereas each Fibonacci-like polynomial of degree n in the interval $(-4, 0)$ has exactly n roots [1], the method of iterations described above enables one to approximate the largest in absolute value root for each polynomial. In order to find other roots, formulas (48) and (49) should be modified. For example, the polynomial $P_4(x) = x^2 + 4x + 3$ has two roots, $x = -3$ and $x = -1$. The former root can be found using formula (49). In order to find the latter root, the equation $x^2 + 4x + 3 = 0$ can be transformed to the form

$$x^2 = \frac{-3x}{x+4}$$

defining the iteration

$$x_{n+1} = -\sqrt{\frac{-3x_n}{x_n + 4}}$$

which converges to -1 for any negative initial value x_1. As the degree of polynomials $P_n(x)$ increases, the appropriate iterative formulas becomes more and more difficult to find if one wants to find all the roots. For example, already in the case of the (third degree) polynomial $P_5(x)$, the roots of the equation $x^3 + 5x^2 + 6x + 1 = 0$ require the application of the following formulas:

$$x_{n+1} = -5 - \frac{6}{x_n} - \frac{1}{{x_n}^2}$$

(a special case of formula (48)),

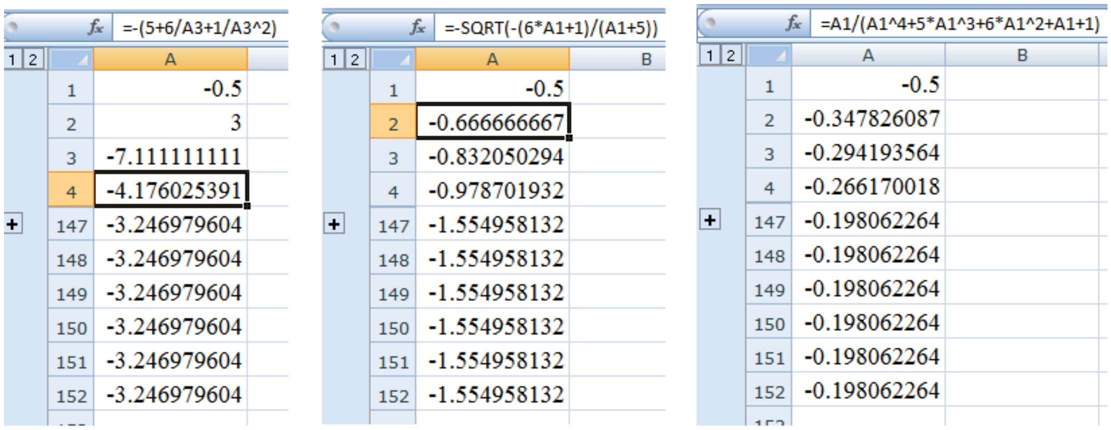
$$x_{n+1} = -\sqrt{-\frac{6x_n + 1}{x_n + 5}}$$

and

$$x_{n+1} = \frac{x_n}{{x_n}^4 + 5{x_n}^3 + 6{x_n}^2 + x_n + 1}$$

It should be noted that different iterative formulas could be used to approximate the roots $x = -3.246979604\ldots$ (Figure 19a), $x = -1.554958132\ldots$ (Figure 19b), and $x = -0.198062264\ldots$ (Figure 19c) of the polynomial $P_5(x)$.

Also, the roots of Fibonacci-like polynomials should be found with the highest precision possible in order to enable the demonstration of the cycle occurring phenomenon. That is why the authors do not use the "goal seek" command from the Tools Menu of Excel because, as mentioned in [3], the method of iterations appears to provide a higher precision in comparison with the "goal seek" command. More details on the use of spreadsheets for the method of iteration can be found in [3].



f_x =-(5+6/A3+1/A3^2)

	A
1	-0.5
2	3
3	-7.111111111
4	-4.176025391
147	-3.246979604
148	-3.246979604
149	-3.246979604
150	-3.246979604
151	-3.246979604
152	-3.246979604

f_x =-SQRT(-(6*A1+1)/(A1+5))

	A	B
1	-0.5	
2	-0.666666667	
3	-0.832050294	
4	-0.978701932	
147	-1.554958132	
148	-1.554958132	
149	-1.554958132	
150	-1.554958132	
151	-1.554958132	
152	-1.554958132	

f_x =A1/(A1^4+5*A1^3+6*A1^2+A1+1)

	A	B
1	-0.5	
2	-0.347826087	
3	-0.294193564	
4	-0.266170018	
147	-0.198062264	
148	-0.198062264	
149	-0.198062264	
150	-0.198062264	
151	-0.198062264	
152	-0.198062264	

Figure 19: (a, b, c). Solving the equation $P_5(x) = 0$ through the method of iterations.

18 The roots of Fibonacci-like polynomials as generators of cycles

Finally, one can use a spreadsheet and data provided by the method of iterations to demonstrate the cyclic behavior of the ratios f_{n+1}/f_n generated by equation (3.5). Put another way, the roots of Fibonacci-like polynomials can be shown to provide the cyclic behavior of generalized golden ratios. As an illustration, we use the data shown in Figure 19. The three roots of the polynomial $P_5(x)$ determine three parabolas in the $(a, b)-$plane of parameters where generalized golden ratios form cycles of period seven. The spreadsheets pictured in Figures 20–22 demonstrate (both numerically and graphically) that each time the straight line $a = 3$ crosses any of the three parabolas

$$\frac{a^2}{b} = -3.246979604$$
$$\frac{a^2}{b} = -1.554958132$$
$$\frac{a^2}{b} = -0.198062264$$

the cycles of period seven occur. Note that the reciprocals $(-3.246979604)^{-1}$, $(-1.554958132)^{-1}$, and $(-0.198062264)^{-1}$ are the roots of the Jacobsthal polynomial $J_6(x) = x_3 + 6x_2 + 5x + 1$.

19 Conclusion

This paper has demonstrated the potential of spreadsheet technology for the discovery of new mathematical knowledge in the context of education, thereby, providing a medium for motivated students to experience mathematics in the making. A classic example of that type is the *Central Limit Theorem* - the unofficial sovereign of probability theory.

B2 f_x =(A2^2)/(-3.246979604)

	A	B	C	D
1	a	b	f_n	f_{n+1}/f_n
2	3	-2.7718068	1	
3			1	1
4			0.22819324	0.228193
5			-2.087227	-9.14675
6			-6.8941886	3.303037
7			-14.897176	2.160831
8			-25.582169	1.71725
9			-35.454415	1.385903
10			-35.454415	1
11			-8.090458	0.228193
12			74.0014124	-9.14675
13			244.429423	3.303037
14			528.170655	2.160831
15			907.000839	1.71725
16			1257.01553	1.385903
17			1257.01553	1

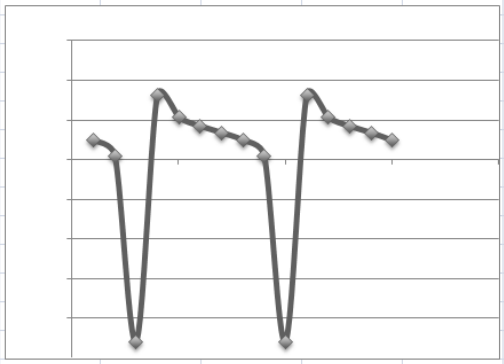


Figure 20: The case $\frac{a^2}{b} = -3.246979604$

B2 f_x =(A2^2)/(-1.554958132)

	A	B	C	D
1	a	b	f_n	f_{n+1}/f_n
2	3	-5.7879372	1	
3			1	1
4			-2.7879372	-2.78794
5			-14.151749	5.076064
6			-26.318841	1.859759
7			2.95291013	-0.1122
8			161.190529	54.58701
9			466.480327	2.893969
10			466.480327	1
11			-1300.5179	-2.78794
12			-6601.5124	5.076064
13			-12277.222	1.859759
14			1377.47449	-0.1122
15			75192.2106	54.58701
16			217603.896	2.893969
17			217603.896	1

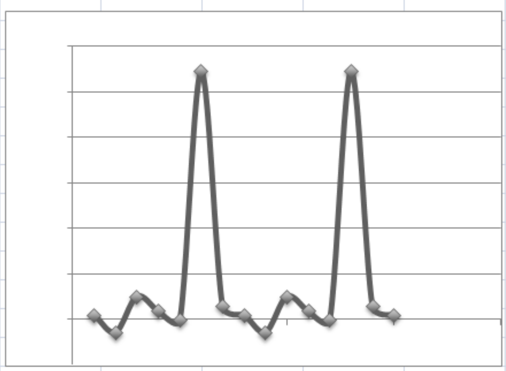
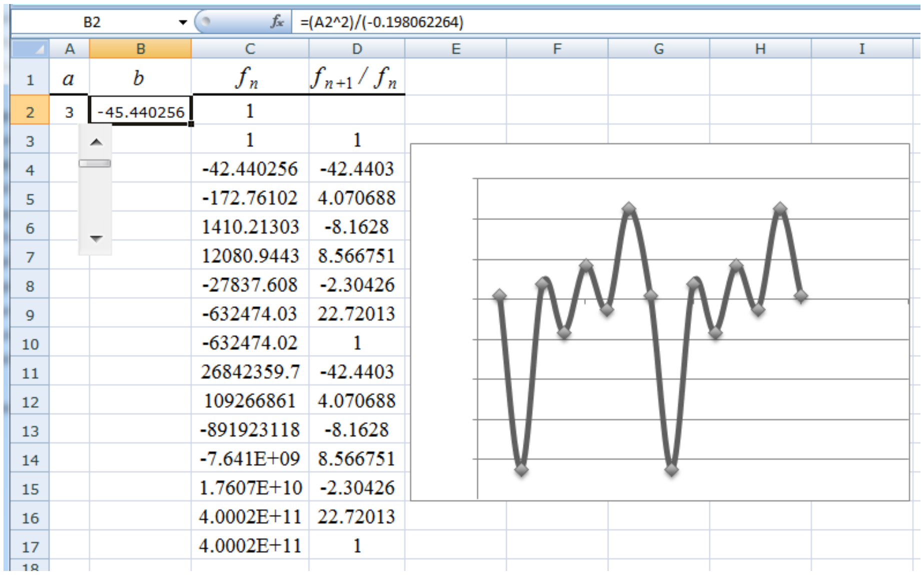


Figure 21: The case $\frac{a^2}{b} = -1.554958132$ yields another 7- cycle ($a = 3$).



B2 =(A2^2)/(-0.198062264)

	A	B	C	D
1	a	b	f_n	f_{n+1}/f_n
2	3	-45.440256	1	
3			1	1
4			-42.440256	-42.4403
5			-172.76102	4.070688
6			1410.21303	-8.1628
7			12080.9443	8.566751
8			-27837.608	-2.30426
9			-632474.03	22.72013
10			-632474.02	1
11			26842359.7	-42.4403
12			109266861	4.070688
13			-891923118	-8.1628
14			-7.641E+09	8.566751
15			1.7607E+10	-2.30426
16			4.0002E+11	22.72013
17			4.0002E+11	1

Figure 22: The case $\frac{a^2}{b} = -0.198062264$ yields yet another 7- cycle ($a = 3$).

In 1901, the outstanding Russian mathematician Aleksandr Lyapunov formulated and proved the theorem in the most general form (allowing random variables to exhibit different distributions) as he was preparing a new course for the students of the University of St. Petersburg [18]. Nowadays, it is the power of technology that can bridge research efforts of mathematics educators and mathematicians. In particular, such a unity of content and pedagogy can enhance the mathematical preparation of prospective teachers who are expected to teach mathematics through a guided discovery mode [5], [13]. Even a small exposure to a mathematical frontier can greatly motivate prospective teachers to teach in accord with the current standards for teaching and recommendations for teachers. Indeed, as is well known, it is difficult to impart the experience of mathematical discovery when teaching mathematics, unless such an experience had been a part of a teacher preparation program [15]. This also includes the teachers' experience with mathematical proofs. In connection with the latter experience, it is important to emphasize the dual role of a spreadsheet in helping learners of mathematics to develop skill in writing proofs. Indeed, the tool is capable of motivating proof (as in the case of the Fibonacci sieve), as well as to support proof (as in the case of identities among Fibonacci-like polynomials). Such a duality is an essential characteristic of a spreadsheet and the inclusion of detailed proofs – both motivated and supported ones – was due in part to the need to highlight this didactic aspect of the use of the tool. By using the body of knowledge associated with Fibonacci numbers, the Golden Ratio, Pascal's triangle, difference equations, and the method of iterations as an example, the paper has demonstrated what topics might be included into the (spreadsheet-enhanced) capstone course sequence for prospective secondary teachers allowing them "to take actions like representing, experimenting, modeling, classifying, and proving" [5, p5]. It is through the actions of that kind that one can appreciate the didactic importance of the joint use of experiment and theory in the development of mathematical concepts. As was emphasized throughout the paper, whereas one needs theory in order to make sense of a spreadsheet-based experiment, one can also benefit from the use of the software as a tool for the validation of the formation of a concept. Ultimately, such a pedagogical approach makes the study of mathematics sufficiently engaging and enjoyable to be used across the whole tertiary mathematics curriculum.

20 Appendix

In this section, details of spreadsheet programming of a number of environments used in this paper for developing ideas about Fibonacci-like polynomials. Below, the notation `(A1)` will be used to present a formula defined in cell `A1`. As always, syntactic versatility of spreadsheets allows for the use of different formulas in constructing both visually and computationally identical environments. In some cases, spreadsheet formulas have already been displayed as part of the figures of this paper in the formula bar of the corresponding spreadsheets.

20.1 Spreadsheet programming for Figure 6 (Fibonacci sieve)

The range of rows 2, 3, and 4 is hidden from view.

`(C2) = 1`,

`(D2) = 1+C2` – replicated to cell `H2`,

`(C3) = 2^C2` – replicated to cell `H3`,

`(C4) = 2^(C2-1)` – replicated to cell `H4`.

`(C5) = SMALL(C6:C104, 2)` – replicated to cell `H5`. This formula collects the smallest number surviving the Fibonacci sieve of orders one (column `C`), two (column `D`), . . . , six (column `H`).

`(C6) = $B6` – replicated to cell `H6`.

`(C7) = IF(MOD($A7 + C$4, c$3) = 0, " ", B7)` – replicated across columns and down rows to cell `H105`.

20.2 Spreadsheet formulas for Figure 7 (use of conditional formatting)

1. Column `B` is filled with Fibonacci numbers and is replicated to column `G`.

2. Highlight the range `C2:C49` (Fibonacci sieve of order one) and open Conditional Formatting dialog box. Condition 1: Formula is `= MOD(A2, 2) > 0`, choose format.

3. Highlight the range `D2:D49` (Fibonacci sieve of order two) and open Conditional Formatting dialog box. Condition 1: Formula is `= OR(MOD(A2, 2)>0,MOD(A2, 4) = 2)`, choose format.

4. Highlight the range E2:E49 (Fibonacci sieve of order three) and open Conditional Formatting dialog box. Condition 1: Formula is = OR(MOD(A2, 2)>0,MOD(A2, 4) = 2,MOD(A2, 8) = 4), choose format.

5. Highlight the range F2:F49 (Fibonacci sieve of order four) and open Conditional Formatting dialog box. Condition 1: Formula is = OR(MOD(A2, 2)>0,MOD(A2,4) = 2,MOD(A2,8) = 4,MOD(A2, 16) = 8), choose format.

6. Highlight the range G2:G49 (Fibonacci sieve of order five) and open Conditional Formatting dialog box. Condition 1: Formula is = OR(MOD(A2, 2)>0,MOD(A2,4) = 2,MOD(A2,8) = 4,MOD(A2, 16)= 8,MOD(A2, 32) = 16), choose format.

20.3 Spreadsheet formulas for Figure 9

1. (D4) = LOOKUP(E10, A4:A444, C4:C444)

2. Cell D4 is given the name a.

3. (D7) = -1; cell D7 is given the name b.

4. Cell D10 is slider controlled, given the name k and assumes values from 0 to 6.

5. (E10) = 2^k;

6. (B1) = 1;

7. (C1) = LOOKUP(E10, A4:A444, B4:B444);

8. (D1) = a*C1 + b*B1 – replicated to cell G1.

9. (G4) = IF(D10 = 0, "", IF(D10 = A5,(a + SQRT(a*a - 4))/2, G4))– replicated to cell G9.

In order to enable the circular reference in the last formula, in Excel Preferences menu, open Calculations dialogue box and check the Limit iterations box (setting 10 iterations maximum). The circular reference makes it possible for already calculated values of the expression

$$\frac{a+\sqrt{a^2-4}}{2}$$

to stay the same as the value of k varies.

20.4 Spreadsheet formulas for Figure 12

1. Range B1:L1 is filled with numbers 0 through 10 and given the name n.

2. Range A2:A21 filled with pairs $(1, 1)$ through $(10, 10)$ and given the name i.

3. (B2) = IF(n >= i, COMBIN(2*n - i, i), " ")- replicated to cell L2.

4. `(B3) = IF(n >= i, COMBIN(2*n -i + 1, i), " ")` – replicated to cell `L3`.
5. The range `B2:L3` is replicated to cell `L21`.
6. `(N2) = SUM(B2:L2)` – replicated to cell `N21`.

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