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Three spreadsheet models of a simple pendulum

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The paper gives three spreadsheet models of simple pendulum motion. In two of them, the graph of the pendulum angle is drawn upon the exact integral solution using numeric integration – the simpler model only uses spreadsheet functions, the general model uses a VBA program. The period results directly from the calculations. In the third model, the period is calculated first using the power series formula. The graph of the pendulum angle is drawn afterwards using Euler's method of solving differential equations. The error in gravity acceleration if calculated upon the standard cosine approximate formula instead of the exact one is graphed.

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Quantities and symbols

α	angle	g	gravity acceleration
β	angle	h	pendulum altitude
ε	computational error in time	m	pendulum mass
φ	pendulum angle	v	pendulum velocity
η	relative error in g	t	Time
ω	pendulum angular velocity	F	Force
Φ	pendulum amplitude	G	gravitational force: $G = gm$
Σ	summation of the power series in the period formula	H	pendulum starting altitude
		L	pendulum length
		T	pendulum period

1. Introduction

Measuring the acceleration of gravity g by simple pendulum (a bob on a weightless cord) is a standard physical experiment in upper secondary schools. Students are acquainted with the fact that the function

$$\varphi(t) = \Phi \cos\left(t\sqrt{g/L}\right), \tag{1}$$

where t is time, φ is the angle, $0 < \Phi \le \pi/2$ is the amplitude, and L is the length of the pendulum (see the table of symbols), is just an approximate one and usable at $\Phi \le 5°$ only, which holds as well for the emerging formulas for the period T and the gravity acceleration g

$$T = 2\pi\sqrt{L/g}\,,\ g = 4\pi^2 L/T^2\ . \tag{2a-2b}$$

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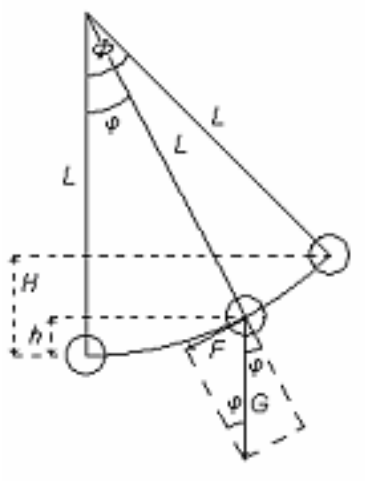


Figure 1: Simple pendulum

There are three topic questions from the students: If $5° < \Phi \leq 90°$: what is the graph of the exact solution like; what are the correct alternatives to formulae (2a, 2b), and how big is the error if one uses formula (2b)?

The answers follow from the exact solution to the pendulum problem. That is given by an analytically incalculable definite integral. Getting a quarter-period of the graph (which enables us to construct the entire period) requires evaluating the integral at each φ from Φ to 0 by a step $\Phi/100$ e.g.; the integral is evaluated a hundred times at each change of Φ, then. Generally, that is impossible without programming. However, the monotonicity of the integrand enables us to pre-calculate the number of subintervals of the main interval for getting a given accuracy. Let ε be the computational error in time. If $0° < \Phi \leq 90°$, $L \leq 10$ m, $1.5 \leq g \leq 20$ ms^{-2}, and $\varepsilon \geq 0.005$ s, then the number of the subintervals comes out less than 170, which allows us to use standard spreadsheet skills i.e. avoid programming. That is the first model. It is simple, and fully interactive. The period is calculated from the quarter-period at $\varphi = 0$. The second model is a general variant of the first one (a bob on a weightless rod) and uses a VBA program. The model works at any physically correct inputs, i.e., $0° < \Phi < 180°$, $L > 0$ m, $g > 0$ ms^{-2}, $\varepsilon > 0$ s, and is only limited by the accuracy of the spreadsheet program. The calculations take some time, thus the model reacts with a delay. The third model is based upon a numeric solution of the governing differential equation. The period is calculated first using the power series formula. The graph of the pendulum angle is drawn afterwards using Euler's method of solving differential equations that is simple to understand. Nevertheless, the solution is impossible without using some tricks that makes it interesting. The model is valid at $0° < \Phi \leq 90°$, $L \leq 100$ m, $0.1 \leq g \leq 100$ ms^{-2}, $\varepsilon \geq 0.00005$ s, and it is quick and fully interactive.

In each model, the exact formula for g follows from the period formula. The error in g if calculated upon the cosine approximate formula (2b) instead of the exact one is computed and graphed at Φ from 0° to 90°.

2. Exact solution to simple pendulum and the exact formula for computing g

Remark: The theory presented in this section can be found e.g. in [1 − 6].
The energy of the pendulum is given by the law of energy conservation (Fig. 1)

$$mgH = mgh + mv^2/2, \tag{3}$$

where m is the mass, v is the velocity, h is the altitude, and H is the starting altitude. Then

$$v = \sqrt{2g(H-h)}. \tag{4}$$

As $H = L - L\cos\Phi$, $h = L - L\cos\varphi$, $v = L\omega = L\,\mathrm{d}\varphi/\mathrm{d}t$ (ω is the angular velocity), then

$$\frac{\mathrm{d}\varphi}{\mathrm{d}t} = -\sqrt{2\frac{g}{L}(\cos\varphi - \cos\Phi)},\ -\Phi \le \varphi \le \Phi,\ \varphi(0) = \Phi. \tag{5}$$

This results in the fundamental formula that gives the inverse function to $\varphi(t)$

$$t = \sqrt{\frac{L}{g}}\int_{\varphi}^{\Phi}\frac{d\alpha}{\sqrt{2(\cos\alpha - \cos\Phi)}},\ -\Phi \le \varphi \le \Phi, \tag{6}$$

where α is angle between Φ and φ. The integral is incalculable analytically. Moreover, it is improper as $\lim_{\alpha\to\Phi}\sqrt{2(\cos\alpha - \cos\Phi)} = 0$, and this excludes numeric integration.
Applying formula $\cos\alpha = 1 - 2\sin^2(\alpha/2)$ and substitution of $\sin(\beta) = \sin(\alpha/2)/\sin(\Phi/2)$, where β is angle, gives

$$t = \sqrt{\frac{L}{g}}\int_{a}^{\pi/2}\frac{d\beta}{\sqrt{1-\sin^2(\Phi/2)\sin^2\beta}}, \tag{7}$$

where $a = \arcsin\left[\sin(\varphi/2)/\sin(\Phi/2)\right]$. This integral is proper as $\sin(\Phi/2) < 1$ at any Φ.

If $\varphi = 0$, then $t = T/4$, so

$$T = 4\sqrt{\frac{L}{g}}\int_{0}^{\pi/2}\frac{d\beta}{\sqrt{1-\sin^2(\Phi/2)\sin^2\beta}}. \tag{8}$$

The integral is the complete elliptic integral of the first kind. The power expansion is

$$T = 2\pi\sqrt{\frac{L}{g}}\left[1 + \left(\frac{1}{2}\right)^2\sin^2(\Phi/2) + \left(\frac{1.3}{2.4}\right)^2\sin^4(\Phi/2) + \ldots\right] = 2\pi\sqrt{\frac{L}{g}}\Sigma, \tag{9}$$

where Σ is for the summation in the brackets. The correct formula for computing g is then

$$g = 4\pi^2 L\Sigma^2 / T^2 \ . \tag{10}$$

Let g_0 be the value computed upon Eq. (2b). The relative error is then

$$\eta = \left| 1 - g_0/g \right| = \left| 1 - 1/\Sigma^2 \right| . \tag{11}$$

We can see that η depend on amplitude Φ but not on length *L*.
We note that Eq. (5) is obtainable by solving the motion equation of pendulum (Fig. 1)

$$m\frac{\mathrm{d}v}{\mathrm{d}t} + mg\sin\varphi = 0 \ , \ \varphi(0) = \Phi \ , \ v(0) = 0 \ . \tag{12}$$

Substitution $v = L\omega = L\,\mathrm{d}\varphi/\mathrm{d}t$ yields the famous pendulum equation

$$\frac{\mathrm{d}^2\varphi}{\mathrm{d}t^2} + \frac{g}{L}\sin\varphi = 0 \ , \ \varphi(0) = \Phi \ , \ \left.\frac{\mathrm{d}\varphi}{\mathrm{d}t}\right|_{t=0} = 0 \ . \tag{13}$$

Using $\frac{\mathrm{d}^2\varphi}{\mathrm{d}t^2} = \frac{\mathrm{d}\omega}{\mathrm{d}t} = \frac{\mathrm{d}\omega}{\mathrm{d}\varphi}\frac{\mathrm{d}\varphi}{\mathrm{d}t} = \frac{\mathrm{d}\omega}{\mathrm{d}t}\omega$ we get

$$\omega\frac{\mathrm{d}\omega}{\mathrm{d}t} + \frac{g}{L}\sin\varphi = 0 \ , \ \omega(\Phi) = 0 \tag{14}$$

with solution

$$\omega = \frac{\mathrm{d}\varphi}{\mathrm{d}t} = -\sqrt{2\frac{g}{L}(\cos\varphi - \cos\Phi)} \ , \ \varphi(0) = \Phi \ . \tag{15}$$

3. Models with using numeric integration

The graph consists of two parts. The first part comprises 108 points (t,φ), where φ goes from Φ to 0 (the first quarter-period) by steps $\Delta\varphi = \Phi/100$ mostly; it is $\Delta\varphi/4$ over the first four points, and it is $\Delta\varphi/2$ over the next eight points to make the model more precise if $\varphi \to \Phi$. The integral is computed at each φ using the trapezoid rule. The minimum number of subintervals for getting a given (small) error ε in *t* is calculable as follows: Let $a = \arcsin[\sin(\varphi/2)/\sin(\Phi/2)]$, $b = \pi/2$, n be the number of subintervals of

interval $\langle a, b\rangle$, $k = \sin(\Phi/2)$, $f(\beta) = 1\big/\sqrt{1 - k^2 \sin^2 \beta}$ be the integrand. Then $\Delta\beta = (b-a)/n$, $\beta_i = \beta_{i-1} + \Delta\beta$, $\beta_0 = a$, $\beta_n = b$, and the trapezoid rule gives

$$t = \sqrt{\frac{L}{g}}\left[\frac{f(a)+f(b)}{2} + \sum_{i=1}^{n-1} f(\beta_i)\right]\Delta\beta. \tag{16}$$

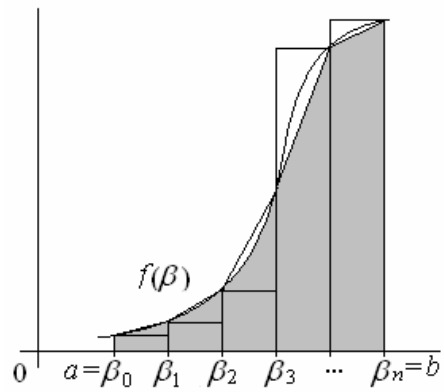


Figure 2: The trapezoid rule and the maximum error

As $f(\beta)$ is ascending (see sheet "Integrand" and Fig. 2), the maximum absolute error ε in time is

$$\varepsilon = \sqrt{\frac{L}{g}}\left[\frac{f(\beta_1)-f(\beta_0)}{2} + \frac{f(\beta_2)-f(\beta_1)}{2} + \cdots + \frac{f(\beta_{n-1})-f(\beta_{n-2})}{2} + \frac{f(\beta_n)-f(\beta_{n-1})}{2}\right]\Delta\beta, \tag{17}$$

then

$$\varepsilon = \sqrt{\frac{L}{g}}\,\frac{f(b)-f(a)}{2}\,\frac{b-a}{n}, \tag{18}$$

which yields

$$n = \sqrt{\frac{L}{g}}\,\frac{f(b)-f(a)}{2}\,\frac{b-a}{\varepsilon}. \tag{19}$$

If $0° < \Phi \le 90°$, $L \le 10$ m, $1.5 \le g \le 20$ ms^{-2}, and $\varepsilon \ge 0.005$ s, then $n < 170$, which allows us to use standard spreadsheet skills only. That is the idea that the first model is based on. The application is in sheet "Model1_IntegralExcel". The main calculations are in the hidden columns N − GH. Function IF is used in columns T − H to match the actual value

of n. The inputs stated above are the limiting ones. The graph of the approximate cosine solution (Eq. 1) is added to make a comparison. The model is in Fig. 3.

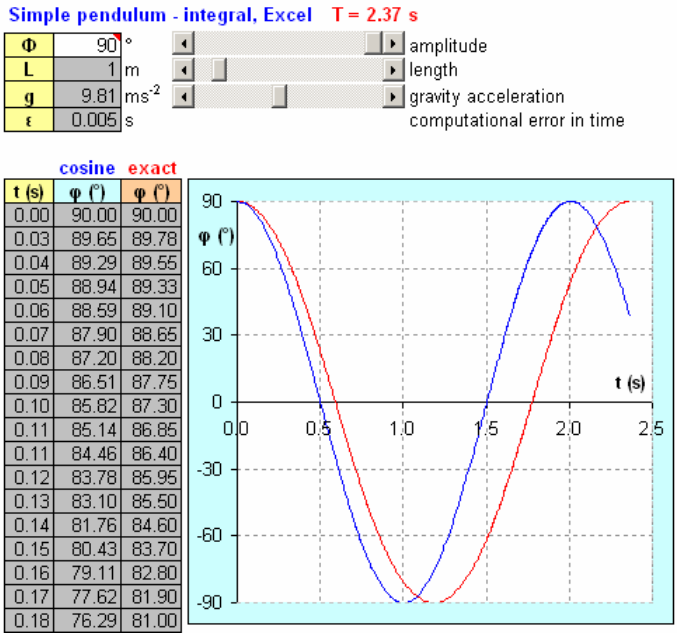
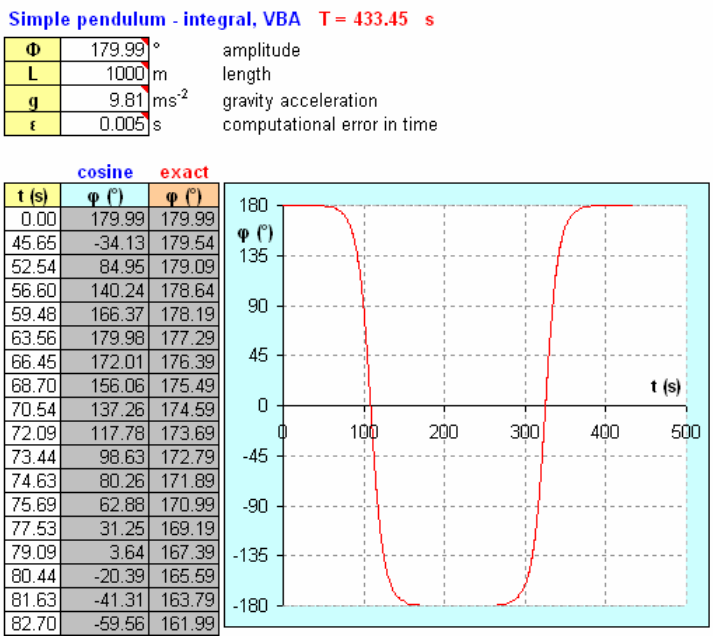


Figure 3: Model 1

The second model is a general variant of the first one and uses a VBA program. The program is a method of a hidden textbox, and it starts at the change of the textbox. The textbox is linked with a hidden cell (columns N, O) that contains the sum of all inputs. Hence, if any input is changed, which is acknowledged by the Enter key, the textbox changes and the program starts. The model works at any physically correct inputs, i.e. $0° < \Phi < 180°$, $L > 0$ m, $g > 0$ ms^{-2}, $\varepsilon > 0$ s, and is only limited by the accuracy of the spreadsheet program. The calculations take some time, thus the model reacts with a delay; that is why no scrollbar is used. The running calculation is indicated by a label "WAIT". If $\Phi = 90°$, $L = 1$ m, $g = 9.81$ ms^{-2}, and $\varepsilon = 0.000005$ s, then the delay is about 3 s; if $\varepsilon = 0.000001$ s, then it is 12 s (at Celeron 1.5 GHz, 512 MB RAM). If $\Phi > 90°$, then the delay increases hugely, thus $\varepsilon \geq 0.005$ s should be used. If $\Phi = 179{,}99°$, $L = 1000$ m, $g = 9.81$ ms^{-2}, and $\varepsilon = 0.005$ s, then the delay is about 55 s. The model with these inputs is in Fig. 4 (the cosine graph is cleared away because there is no point in drawing that at such amplitude as almost seven roughly drawn cosine periods come out within the exact one). The application is in sheet "Model2_IntegralVBA". The program (with comments) is accessible by double-click on the textbox.



Simple pendulum - integral, VBA T = 433.45 s

Φ	179.99	°	amplitude
L	1000	m	length
g	9.81	ms^{-2}	gravity acceleration
ε	0.005	s	computational error in time

t (s)	cosine φ (°)	exact φ (°)
0.00	179.99	179.99
45.65	-34.13	179.54
52.54	84.95	179.09
56.60	140.24	178.64
59.48	166.37	178.19
63.56	179.98	177.29
66.45	172.01	176.39
68.70	156.06	175.49
70.54	137.26	174.59
72.09	117.78	173.69
73.44	98.63	172.79
74.63	80.26	171.89
75.69	62.88	170.99
77.53	31.25	169.19
79.09	3.64	167.39
80.44	-20.39	165.59
81.63	-41.31	163.79
82.70	-59.56	161.99

Figure 4: Model 2

The second part of the graphs in both models comprises 321 points (t,φ), where φ goes from 0 to $-\Phi$ (the second quarter-period), then φ goes back from $-\Phi$ to Φ (the second half-period), and time t is computed upon the symmetry of the graph using functions INDIRECT and ADRESS. The sheets are locked with no password. The white cells are accessible only. The formulae are hidden under a white rectangle below the graph.

4. The model with using Euler's method of solving differential equations

Substituting $k = \sin(\Phi/2)$ into Eq. (9) gives

$$T = T_0\left[1+\left(\frac{1}{2}\right)^2 k^2 + \left(\frac{1.3}{2.4}\right)^2 k^4 + \left(\frac{1.3.5}{2.4.6}\right)^2 k^6 + ..\right].. \tag{20}$$

The series

$$T_{\text{major}} = T_0\left[1+\frac{1}{2}k^2 + \frac{1.3}{2.4}k^4 + \frac{1.3.5}{2.4.6}k^6 + ..\right] = T_0\Big/\sqrt{1-k^2} \tag{21}$$

is a majorant one to Eq. (20) i.e. each term is bigger than the corresponding term in Eq. (20). If we take the first n terms, then the remainder (i.e. the error) is

$$\varepsilon = T_0\left[1\Big/\sqrt{1-k^2} - \left(1 + \frac{1}{2}k^2 + \frac{1.3}{2.4}k^4 + \frac{1.3.5}{2.4.6}k^6 + ..\right)\right] \tag{22}$$

This is sure to be bigger then the corresponding remainder of period *T*, i.e. the error that we get if we take the first *n* terms in Eq. (20) is sure to be less than ε. The spreadsheet application that enables us to find the minimum *n* is in sheet "Minimum N for T". It results that *n* = 23 is enough if $0° < \Phi \le 90°$, $L \le 100$ m, $0.1 \le g \le 100$ ms^{-2}, and $\varepsilon \ge 0.00005$ s. The redundant terms of the given *n* = 25 ones are in red numbers.

The model uses Euler's method of solving differential equations for drawing the graph at $0 \le t \le T$. Replacing the differentials with differences in Eq. (5) gives

$$\Delta\varphi = -\sqrt{2\frac{g}{L}(\cos\varphi - \cos\Phi)}\,\Delta t\,,\ \varphi(0) = \Phi\,. \tag{23}$$

If φ goes from Φ to $-\Phi$, then $\varphi_i = \varphi_{i-1} - \Delta\varphi$, $\varphi_0 = \Phi$, *i* = 1, 2, … *k*, where *k* is the number of subintervals Δt of period *T*, i.e. $\Delta t = T/k$. If φ goes back from $-\Phi$ to Φ, then $\varphi_i = \varphi_{i-1} + \Delta\varphi$.

The application is in sheet "Model3_Euler". The change from – to + in the formula for φ_i is realized by the IF function nested twice (hidden column Q). The problem with the zero derivative of φ at *t* = 0 ($\varphi = \Phi$ then) that causes $\varphi_i = 0$ at any *i* is settled with a trick: if (the IF function) the square root in Eq. (23) gives zero, then we put $\Delta\varphi = 10^{-15}$ (the lowest possible nonzero value in a spreadsheet cell) or else $\Delta\varphi$ is given by Eq. (23) (column S). The exact period is calculated in range N12:O39. Despite the simplicity of the method, *k* = 2000 is enough for getting $\varphi = 89.99952°$ at $t = T$ if the inputs are $\Phi = 90°$, $L \le 100$ m, $0.1 \le g \le 20$ ms^{-2}, and $\varepsilon \ge 0.00005$ s, which are the limiting inputs for the model. It is locked without a password. The formulae are hidden under a white rectangle below the graph. The model is in Fig. 5.

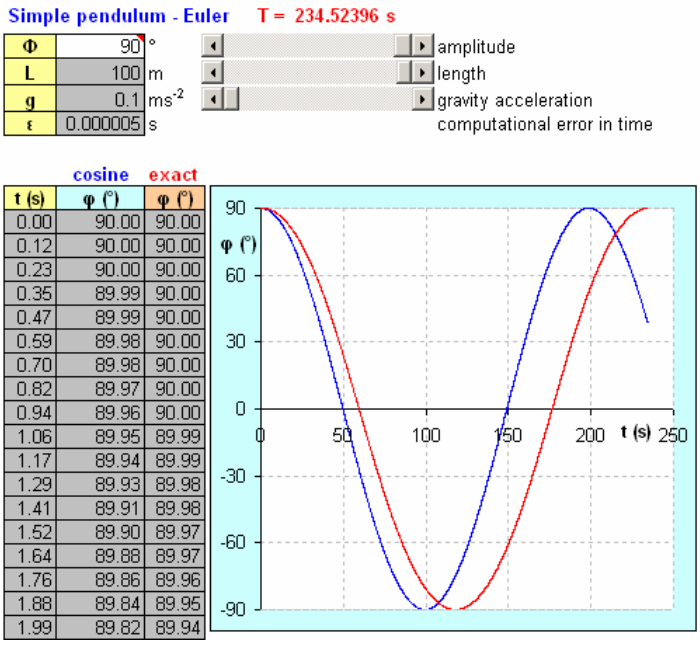


Figure 5: Model 3

The application that computes the error η in g (Eq. 11) at various Φ and draws the $\eta(\Phi)$ graph is in sheet "Error in g". The exact period is computed upon Eq. (20) using the first $n = 25$ terms. If $\Phi = (°)$ 5, 16, 36, 52, 90, then $\eta = (\%)$ 0.095 (<0.1), 0.97 (<1), 4.86 (<5), 9.99 (<10), 28.2, respectively. The $\eta(\Phi)$ graph is in Fig. 6.

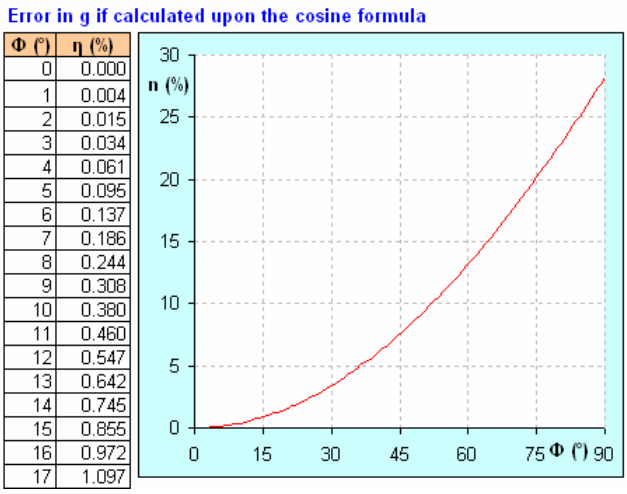


Figure 6: Error in g

5. Conclusions

The simple pendulum is involved in the physics curriculum in higher secondary school (age 15 – 19) usually, where measurements of g upon formula (2b) are carried out.

One way of using the models is just to show them to the students and study the behaviour at various inputs, especially to show that the cosine and exact graphs merge into one at small amplitudes, and that formula (2b) is absolutely unusable at $\Phi > 16°$ as the error is bigger than 1 %. Instead, formula (10) has to be used, where the period T is measured by a stopwatch, and the sum is computed (e.g. like in sheet "Error in g").

Another way is to build up a model with students interested in physics and mathematics who have good spreadsheet skills. The model with the VBA program is an example of programming mathematical tasks, and it is meant for students who are interested in programming. The program is easily transferable to other developing environments (the author has developed a Borland Delphi version, e.g.). However, the two other "pure spreadsheet" models are quick and precise enough, and the input ranges exceed by far the needs of school practice.

Quantity	**Symbol (Unit)**	**Model 1**	**Model 2**	**Model 3**
Amplitude	Φ (°)	0 – 90	0 – 180	0 – 90
Length	L (m)	0.001 – 10	> 0	0.001 – 100
Gravity acceleration	g (ms^{-2})	1.5 – 20	> 0	0.1 – 100
Computational error in time	ε (s)	0.005	> 0	0.000005

Table 1: Summarization of the models

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