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Extensive games with time structures

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Abstract

We introduce personal time-structures and common time-structures to extensive games. These structures restrict the class of extensive games of Kuhn [8]. We show that if a player has perfect recall, then the game is personally time-structured for him. In the other direction, there are many personally time-structured games that do not satisfy perfect recall. The only known condition on memory that is required by a personally time-structured game is that no player is absent-minded. Common time-structures, on the other hand, are not related to the perfect recall condition at all. An extensive game may have perfect recall and yet no common time-structure, and conversly, a game may have a common time-structure, and no player has perfect recall. Common time-structures are used to extend backward induction results to games with imperfect recall.

1 Introduction

We introduce the notions of personal time-structures and common-time structures to extensive games. To motivate these structures, we start with the game of Figure 1 played by a lecturer, Tom, and a graduate student, Andy. While this game is one of perfect recall, Tom has some difficulty reasoning through it since his reasoning about Andy takes him "back to the future".

The game is interpreted as follows. Andy will be informed of some good news about a job prospect. Tom will also be informed independently of Andy. Chance decides whether Andy or Tom is informed "first". Once Andy is informed, he will either make "other plans" or "stick around" to

*I thank Yukihiko Funaki for suggesting that time consistency must be difficult to discuss in a game without a clearly defined time structure. I thank Mamoru Kaneko for encouraging me to pursue time structures in extensive games.

celebrate. Once Tom is informed, he must decide either to arrange an "early party" or a "late party" for Andy. The problem for Tom, is that he should not be the one revealing the news to Andy, and hence an "early party" before Andy has been informed, would be a disaster. On the other hand, an "early party" when Andy has already been informed would be a great success. Andy, of course, wants the party as soon as possible after he has been informed, and if he "sticks around" and has to wait, he wishes he had made "other plans".

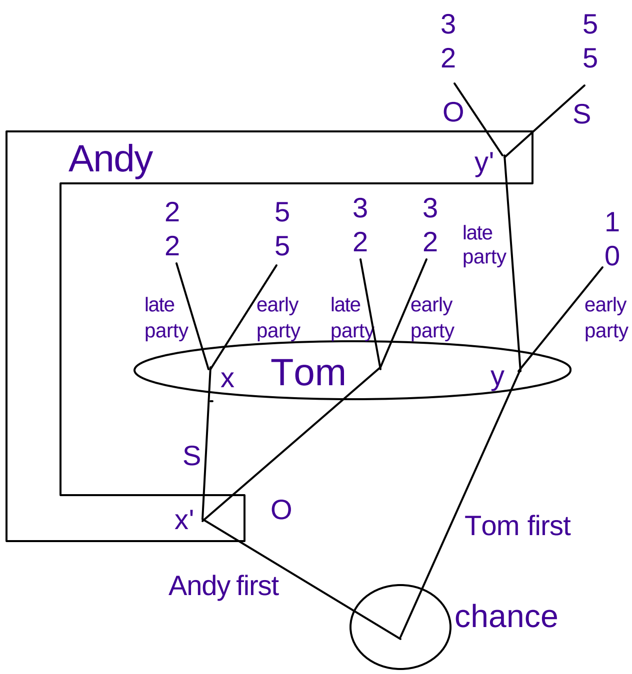

Figure 1: Back to the Future

We give Andy's payo¤ at each outcome above Tom's. For example, if Andy chooses "other plans", then he gets 3 and Tom gets 2, regardless of Tom's decision.

While this game has the strange feature that one information set wraps itself around another, it is an extensive game of perfect recall as de…ned by Kuhn [8].

If Tom tries to reason through the game, he might notice that sometimes Andy decides before him. If he can …nd what Andy has likely done in the past, then this might help him to decide what to do. Suppose he looks to the past and puts himself in Andy's shoes. This type of argument is standard in game theory. The minute he puts himself in Andy's shoes in the past, however, he …nds himself in the future[1] since Andy does not distinguish

[1] Mamoru Kaneko pointed out the "back to the future" nature of this situation.

between x' and y', and y' occurs in the future for Tom. Moving further back in time, Tom finds himself in the present, then the past, then the future and so on. Undoubtedly, in these circumstances, Tom might find it difficult to come to a conclusion about when to throw the party.

The "back to the future" problem encountered by Tom, came about because he could not determine if Andy moves before him or after him. We will find in the upcoming sections of this paper that such problems can be avoided by imposing a common time-structure on the game. We discuss both common time-structures and personal time-structures.

In Section 3.1 of this paper we introduce the notion of a personal clock and use it to define a personally time-structured game. If a game is personally time-structured for a player, then the player can order his own decisions across time. The main result of that section is that a game is personally time-structured for player i if and only if the relation $\prec$ describing precedence over his information sets is acyclic (Theorem 3.1).

Personally time-structured games are found to be related to conditions on memory as described by a player's information partition in an extensive game. These relationships are discussed in Section 3.2. In Corollary 3.4 we show that if a player's information partition satisfies perfect recall, or even a weaker form of recall known as occurrence memory, then the game is personally time-structured for him. In the other direction, there are many personally time-structured games that do not satisfy even perfect recall. The only implication of personally time-structured games for known conditions on memory is that absent-mindedness as defined by Piccione and Rubinstein [11] is not allowed.

In many game theoretic situations, like the game of Figure 1, ordering one's own decisions with the decisions of others is also important. In Section 4.1 we introduce the notion of a common clock and use it to define a commonly time-structured game. In a commonly time-structured game, players agree on the ordering of decisions of all players across time. We find that a game is commonly time-structured if and only if the relation $\prec$ describing precedence over all information sets of all players in the game is acyclic (Theorem 4.1). This result is analogous to the result of Theorem 3.1 for personal clocks.

When it comes to memory, however, we do not get analogous results for common clocks. For example, the analogous result to Corollary 3.4 for common clocks does not hold. There are games, like Figure 1, where every player has occurrence memory and even perfect recall, but the game is not commonly time-structured.

The notion of a common clock brings up the possibility of players moving simultaneously. We discuss simultaneous moves in Section 4.2 and prove a modified version of Theorem 4.1 for such games.

Many economic and game theoretic situations are commonly time-structured or at least personally time-structured. Both of these structures enable the players and game theorists to reason more clearly in games of imperfect recall.

In Section 5 we make use of common time-structures to allow players to apply "backward induction" reasoning to a game. This type of reasoning is often associated with the doctrine that "bygones are bygones." One might conjecture that such a doctrine has nothing to do with memory. In support of this conjecture, we show that backward induction reasoning can be applied to some games of imperfect recall (Theorem 5.1).

2 Finite Extensive Games

In this paper we exploit various conditions on a binary relation on a set of information sets and a set of nodes. Therefore, we first summarize, in section 2.1, the definitions and restrictions for those relations. In section 2.2 we define finite extensive games.

2.1 Preliminaries

We adopt the convention, whenever possible, of using upper case letters like U to denote a set and lower case letters like u to denote an element in a set.

A binary relation $\succeq$ in a set K is a subset of $K \times K$. We write $x \succeq y$ iff $(x, y) \in \succeq$, and $x \not\succeq y$ iff $(x, y) \notin \succeq$. An element $x_0 \in K$ is a smallest element in K iff $x_0 \succeq y$ for all $y \in K$. An element $x \in K$ is a minimal element in K iff for all $y \in K$, $y \succeq x$ implies $x \succeq y$.

Let $\succeq$ be a binary relation in a set K. The relation $\succeq$ is:

(a) anti-symmetric iff for all $x, y \in K$, $x \succeq y$ and $y \succeq x$ imply $x = y$.
(b) transitive iff for all $x, y, z \in K$, $x \succeq y$ and $y \succeq z$ imply $x \succeq z$.
(c) complete iff $x \succeq y$ or $y \succeq x$ for all $x, y \in K$.
(d) a partial ordering iff $\succeq$ is transitive and anti-symmetric.
(e) a complete ordering iff $\succeq$ is transitive, anti-symmetric, and complete.

Observe that the anti-symmetry of $\succeq$ implies the uniqueness of a smallest element.

We obtain the strict binary relation $\succ$ from the binary relation $\succeq$ by defining $x \succ y$ iff $x \succeq y$ and $y \not\succeq x$. We also write $x \not\succ y$ iff not $(x \succ y)$.

Since we focus on the strict binary relation $\succ$ obtained from a binary relation $\succeq$ on K, we give the corresponding relevant conditions for $\succ$. The strict binary relation $\succ$ is:

(a′) asymmetric iff for all $x, y \in K$, $x \succ y$ implies $y \not\succ x$.
(b′) transitive iff for all $x, y, z \in K$, $x \succ y$ and $y \succ z$ imply $x \succ z$.
(g′) irreflexive iff $x \not\succ x$ for all $x \in K$.

Observe that $\succeq$ is a partial ordering if and only if $\succ$ obtained from $\succeq$ is asymmetric and transitive. Therefore, we also say that $\succ$ is a partial ordering in such a case. Using Á, the de…nition of a minimal element x is that y § x for all y 2 K.

2.2 Finite Extensive Game

In de…ning a …nite extensive game, we follow most closely the de…nition of Selten [13] which is based on Kuhn [8] and indirectly on von Neumann and Morgenstern [14]. The main distinction between our de…nition and those of Kuhn [8] and Selten [13] is the conditions imposed on the information pattern.

A …nite extensive game ¡ is a sextuple ((K; ¹); P; U; C; p; h) de…ned for a …nite set of players f0; 1; ::::; ng. The chance player (nature) is player 0, and N = f1; ::::; ng is the set of personal players.

The elements of ¡ are de…ned as follows.

(1) (K; ¹) is a …nite tree, which means:

(1.a) K is a …nite set of nodes partially ordered by the binary relation ¹,
(1.b) for each x 2 K, fy 2 K : y ¹ xg is completely ordered by ¹, and
(1.c) K has the smallest element x_0 called the root node.

The strict predecessor relation Á for K is derived from ¹ in the way described in Section 2.1. Most of our analysis focuses on conditions for this strict predecessor relation. We say that x is a predecessor of y whenever x Á y and that x is an immediate predecessor of y whenever x Á y and there is no y^0 2 K with x Á y^0 and y^0 Á y. When x is a predecessor of y, we say that y is a successor of x.

The set K is partitioned into the set of decision nodes X, those with at least one successor, and the set of endnodes Z, those without successors. Typical elements of X will be denoted by x or y. Typical elements of Z will be denoted by z or z^0. The standard game tree diagram is obtained from (K; ¹) by drawing an edge (or branch) between each node and each of it's immediate successors.[2]

(2) P is a player partition, which means P is a partition of X into n + 1 player sets $P_0; P_1; :::::; P_n$, some of which are possibly empty. Recall that P is a partition of a set X i¤ P is a disjoint collection of subsets of X whose union is K. The nodes in P_i constitute the positions in the game where player i, and only player i, may be called upon to make a decision.

(3) U = f$U_0; U_1; ::::; U_n$g is an information pattern, which means:

(3.a) U_i is a partition of P_i for each player i in f0; 1; :::; ng,
(3.b) for each u 2 U_0, u is a singleton, and

[2] Often analysis is restricted to games trees with non-trivial decision nodes, that is, those with more than one immediate successor. We do not include this restriction here.

(3.c) for each U_i and $i \neq 0$, if $u \in U_i$ and x and y are nodes in u, then the number of immediate successors of x equals the number of immediate successors of y.

Each U_i is called player i's information partition and an element $u \in U_i$ is called an information set of player i. Condition (3.b) is that chance moves are independent of each other. Condition (3.c) is based on the idea that a player should not be able to distinguish between two nodes in the same information set.

Kuhn [8] also put further restrictions on each U_i. We will discuss Kuhn's restrictions and others in later sections of this paper.

(4) $C = \bigcup_{u \in U} C_u$ is a choice partition, which means for each $C_u \in C$:

(4.a) C_u is a partition of A_u, where A_u is the set of immediate successors of nodes in u, and
(4.b) for all $c \in C_u$, c contains exactly one immediate successor of each $x \in u$.

The elements $c \in C_u$ are called choices at u and typically used to name branches in the tree.

(5) p is a completely mixed probability assignment over chance moves which means $p : \bigcup_{u \in U_0} C_u \to \mathbb{R}$, $p(c) > 0$ for all $c \in \bigcup_{u \in U_0} C_u$, and $\sum_{c \in C_u} p(c) = 1$ for all $u \in U_0$.

(6) h is a payoff function which means $h : Z \to \mathbb{R}^n$. For each $z \in Z$, $h(z) = (h_1(z), \ldots, h_n(z))$ gives the payoffs to the personal players if node z is reached.

This concludes our description of a finite extensive game. Observe that it is given independently of any strategy or solution concept. We will introduce strategies and solution concepts in Section 5 when we discuss some implications of time-structured games for equilibrium analysis. Until that time, we will focus on the structure of the game and in particular, the time structure.

3 Personal Time-Structures

In this section we consider personal time-structures and personal clocks. A personal clock allows a player to order his own information sets (moves) across time by assigning a time to each information set (move) so that later moves occur at a later time. In Section 3.1, we define personal clocks and give a necessary and sufficient condition in terms of $\prec$ for a game to be personally time-structured. In Section 3.2 we discuss the relationship between personal time-structures and memory.

3.1 Personal Clocks

A natural foundation for the ordering of information sets is the relation Á de…ned on K. We thus extend Á to the information partition U_i of personal player i in the following way.

Extension of Á to U_i. For u; v 2 U_i we write u Á v i¤ x Á y for some x 2 u and some y 2 v.

Some readers may think this de…nition is too weak. One alternative is to de…ne u Á v i¤ for each y 2 v there exists an x 2 u such that x Á y. Unfortunately, this alternative does not capture our intentions as is shown near the end of Section 3.2.

We can now de…ne a personal clock.

Personal Clock. A personal clock T_i of a player i is a natural number-valued function on U_i such that for all u; v 2 U_i,

$$u \text{ Á } v \text{ implies } T_i(u) < T_i(v): \quad (3.1)$$

Some remarks on this de…nition are in order. A clock is an ordinal concept in the sense that any natural number-valued monotone transformation of T_i is also a clock. Another remark is on the one directional implication from u Á v to $T_i(u) < T_i(v)$. We do not require the converse of it, since information sets may not be comparable by Á, even though one occurs at a later time. We can require the converse for any pair u; v 2 U_i which are comparable by Á.

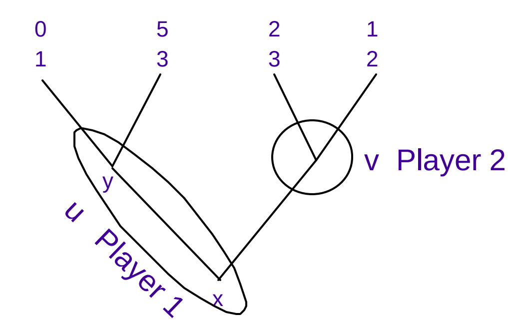

Figure 2

In many economic and game theoretic situations, time is already part of the description. In such cases, it might be straightforward to …nd a personal clock for each player. However, extensive games are not de…ned in terms of time, and thus it may be impossible to …nd a personal clock for some players in some extensive games.

The game of Figure 2 is one such example. It is a two-player game with U_1 = fug and U_2 = fvg. We can assign a personal clock to player 2, for example, $T_2(v) = 1$. However, we cannot find a personal clock for player 1. Since $x \succ y$ and $x, y \in u$ we have $u \succ u$ and thus (3.1) cannot be satisfied by any T_1.

The game of Figure 1 played by Andy and Tom is an example where we can find a personal clock for each player. In fact, whenever a player i has only one information set, there is a personal clock for player i, as long as the relation $\succ$ is irreflexive in U_i.

If Figure 1 is treated as a one-player game with two information sets, i.e., $U_1 = \{Andy, Tom\}$, then it does not have a personal clock. We have Andy $\succ$ Tom and Tom $\succ$ Andy and thus no T_1 can be found satisfying (3.1).

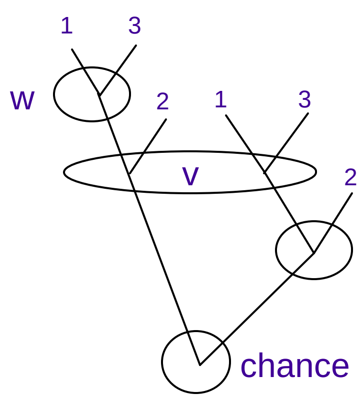

Figure 3

So far we have seen that a personal clock does not exist when the relation of $\succ$ is asymmetric as in the case of Figure 2, or irreflexive as in the case when Figure 1 is treated as a one-player game. The precise limit for when a personal clock can be found for a player, can be stated in terms of a condition on $\succ$ known as acyclicity.[3]

Acyclicity: The strict relation $\succ$ is acyclic in a set U_i iff for any finite sequence of elements $u_1, \ldots, u_k$ of U_i, if $u_1 \succ u_2 \succ \cdots \succ u_k$ then $u_k \nsucc u_1$.

Some remarks on acyclicity are in order. Acyclicity implies both asymmetry and irreflexivity. It is weaker than transitivity, as the one-player game of Figure 3 with $U_1 = \{u, v, w\}$ shows. While $\succ$ is acyclic on U_1 in Figure 3, it is not transitive since $u \succ v$ and $v \succ w$, but $u \nsucc w$. Transitivity and irreflexivity together imply acyclicity. An equivalent definition of acyclicity is that each subset of U_i has a minimal element.

[3] The notion of acyclicity has been discussed in other contexts within the social sciences. For example, von Neumann and Morgenstern (1944, Chapter XII, pages 589-603) discuss acyclicity as a potential property of a solution concept like their dominance relation. Sen (1979, pages 15 and 47) discusses acyclicity of preference relations.

The next theorem shows that acyclicity of $\succ$ in a player's information partition U_i is the dividing line for when an extensive game can be treated as having a personal clock for player i. When a personal clock for player i can be found in a game Γ, we say that Γ is personally time-structured for player i.

Theorem 3.1 (Personal Clock) Let Γ be an extensive game with information pattern $U = \{U_0; U_1; \ldots; U_n\}$. The following two statements are equivalent.
(1) The strict binary relation $\succ$ in U_i is acyclic.
(2) Γ is personally time-structured for player i.

Proof: (1) implies (2). Suppose that $\succ$ is acyclic in U_i. Let $u \in U_i$ be arbitrarily given. Assign the time $T_i(u) = \max\{k : u_1; \ldots; u_k$ is a finite sequence of elements of U_i with $u_1 \succ u_2 \succ \cdots \succ u_k$ and $u_k = u\}$. Since the game is finite and $\succ$ is acyclic in U_i, $T_i(u)$ is a natural number. Since u was arbitrarily given from U_i, the same procedure can be used to assign, to each $v \in U_i$, a natural number $T_i(v)$.

Let $u; v \in U_i$ be arbitrarily given. Suppose $u \succ v$. We need to show that (3.1) is satisfied, i.e., that $T_i(u) < T_i(v)$. Since $T(u) = n$ for some natural number n, there is a finite sequence $u_1; \ldots; u_n$ of n elements of U_i such that $u_1 \succ u_2 \succ \cdots \succ u_n$ and $u_n = u$. Since $u \succ v$, we find a finite sequence $u_1; \ldots; u_{n+1}$ of $n+1$ elements of U_i such that $u_1 \succ u_2 \succ \cdots \succ u_{n+1}$, $u_n = u$, and $u_{n+1} = v$. By definition that $T_i(v)$ is the maximum number of i's information sets up to v, we have $T_i(v) \geq n + 1$.

(2) implies (1). Suppose that $\succ$ is not acyclic in U_i. Then there is a finite sequence of elements of U_i with $u_1 \succ u_2 \succ \cdots \succ u_k$ and $u_k \succ u_1$. But then (3.1) requires $T_i(u_1) < T_i(u_1)$, which is impossible. $\square$

A game need not be personally time-structured for all players. For example, the game of Figure 2 is personally time-structured for player 2, but not for player 1.

Perhaps the implication of (2) by (1) in Theorem 3.1 is the most surprising part. The reader may think that acyclicity is too weak to obtain it. It may help the sceptical reader to note that if we restrict attention to a subset U_i' of U_i such that all elements of U_i' are comparable, then acyclicity of $\succ$ in U_i implies $\succ$ is a complete ordering in U_i'.

3.2 Personal Time-Structures and Memory

In this section we discuss the relationship between personally time-structured games and conditions on memory of a player as described by their information partition. We show that a condition memory weaker than perfect recall, known as occurrence memory, implies the game is personally time-structured. Since every player with perfect recall has occurrence memory, the game is personally time-structured for every player with perfect recall. However, not all personally time-structured games involve players with perfect recall, or even some much weaker forms of recall known in the literature of extensive games.

The only implication of personally time-structured games for conditions on memory, is that absent-mindedness, as defined by Piccione and Rubinstein [11], is ruled out. This finding allows us to consider quite a large class of games with imperfect memory within the context of personally time-structured games.

To discuss conditions on memory, we extend the relation $\succ$ on K to nodes, choices, and information sets. For $u \in U_i$ and $x \in P_i$, we write $u \succ x$ iff there is some $y \in u$ satisfying $y \succ x$. For $x, y \in P_i$, and $c \in C$, we write $x \succ_c y$ iff $x \succ y$ and c is the choice taken at x to get to y. Finally, for $u \in U_i$, $c \in C$, and $x \in P_i$, we write $u \succ_c x$ iff there is some $y \in u$ satisfying $y \succ_c x$.

Kuhn ([8], Definition 2 (II), page 195) gave the following condition as part of his definition of an extensive game.

Conscious-minded: The information partition U_i of a player $i \in N$ satisfies the conscious-minded condition iff the strict binary relation $\succ$ in U_i is irreflexive.

Player 1, in the game of Figure 2, does not satisfy this condition. The negation of conscious-mindedness is called absent-mindedness [11], and thus player 1 in Figure 2 may be called absent-minded. Most all games discussed in the economics and game theory literature satisfy the conscious-minded condition. We have the following corollary which is an application of Theorem 3.1.

Corollary 3.2 *Let Γ be an extensive game with the information pattern $U = \{U_0, U_1, \ldots, U_n\}$. If Γ is personally time structured for personal player i, then U_i satisfies the conscious-minded condition.*

Since Kuhn's definition of an extensive game is equivalent to ours except for his restriction of conscious-mindedness, we find that every personally time-structured game for each player is an extensive game according to Kuhn [8]. We note also that Corollary 3.2 is equivalent to the proposition that a player cannot be absent-minded in a game that is personally time-structured for him.

Since absent-mindedness was used by Piccione and Rubinstein [11] in a one-player game to show a potential distinction between ex ante optimality and time-consistency of a strategy, one might wonder if such distinctions disappear in personally time-structured games. Kline [4] showed these distinctions can arise whenever a condition on memory known as "a-loss recall" is violated. This condition is discussed later in this section where we find that violations of it occur in personally time-structured games.

Kuhn [8] also introduced the perfect recall condition to extensive games. It is interpreted as requiring a player to recall both what he did and what he observed.

Perfect Recall: The information partition U_i of player i has perfect recall iff for all $u, v \in U_i$, all $x, y \in u$, and all $c \in C_v$, $v \succ_c x$ implies $v \succ_c y$.

It turns out that if a player has perfect recall, then the strict binary relation $\succ$ on U_i is not only acyclic, but it is also a partial ordering. The partial ordering property holds for a weaker condition on memory known as occurrence memory [10]. This was shown by Ritzberger ([12], Lemma 1). The condition of occurrence memory is interpreted as requiring a player to recall what he observed, though he might forget what he did.

Occurrence Memory: The information partition U_i of a player i satisfies occurrence memory iff for all $u, v \in U_i$; and all $x, y \in u$, if $v \succ x$ then $v \succ y$.

Every player with perfect recall has occurrence memory since occurrence memory is obtained from perfect recall by only removing the c from $\succ_c$. For the sake of making this paper a more self-contained one, we give a proof of the following Lemma.

Lemma 3.3 (Ritzberger [12]) If U_i satisfies occurrence memory, then $\succ$ is a partial ordering in U_i.

Proof: (i) $\succ$ is transitive in U_i. Suppose $u, v, w \in U_i$ are given with $u \succ v$ and $v \succ w$. Consider an arbitrary $x \in v$. Then by occurrence memory and $u \succ v$, there is a $x' \in u$ with $x' \succ x$. Similarly, by occurrence memory and $v \succ w$, there is $y \in w$ with $x \succ y$. Hence, by transitivity of $\succ$ on K we have $x' \succ y$, a fortiori, $u \succ w$.

(ii) $\succ$ is asymmetric in U_i. Suppose not. Then there are distinct $u, v \in U_i$ with $u \succ v$ and $v \succ u$. But then by (i) just proved, transitivity implies $u \succ u$. Since $\succ$ is a partial ordering in K it is also acyclic. Since the tree is finite, the set of nodes $\{x \in K : x \in u\}$ is a finite set and thus has a minimal element by acyclicity. Call this element y. Since y is minimal in u, there is no $x \in u$ with $x \succ y$ and occurrence memory is violated. $\blacksquare$

By Lemma 3.3 and Theorem 3.1 we get the result of Corollary 3.4 that a game with occurrence memory for player i is personally time-structured for player i.

Corollary 3.4 Let Γ be an extensive game with the information pattern $U = \{U_0, U_1, \ldots, U_n\}$. If U_i satisfies occurrence memory for player i, then Γ is personally time-structured for player i.

The converse of Corollary 3.4 is not true as is demonstrated by the one-player game of Figure 3. That game was shown to be personally time-structured, but Á is not a partial order in U_1 because it is not transitive.

One natural question is whether or not occurrence memory characterizes the information partitions that are partially ordered by Á. The answer is no. The game of Figure 4 is a one-player game with U_1 = fu; vg. The relation Á completely orders U_1, but U_1 doesn't satisfy occurrence memory since u 2 U_i, x; y 2 u, and v Á x, but v § y.

In light of this …nding, we might now ask if there are other known conditions on memory for a player i that are both weaker than occurrence memory and imply Á is a partial ordering in U_i. One weaker condition on memory is known as "a-loss recall" (Kaneko and Kline [3]). This condition allows a player to forget things he did as well as to forget things he observed.

A-loss Recall The information partition U_i satis…es a-loss recall i¤ for all u; v 2 U_i, all x; y 2 u, and all c 2 C_v, if v $Á_c$ x, then either: (1) v $Á_c$ y or (2) there exists w 2 U_i and distinct d; e 2 C_w satisfying w $Á_d$ x and w $Á_e$ y.

Possibility (1) of this de…nition is just perfect recall. (2) is interpreted as requiring that each loss of memory of a player can be traced back to some loss of memory of his own actions. Kaneko and Kline [3] and Kline [4] showed that a-loss recall and occurrence memory share some similar properties.

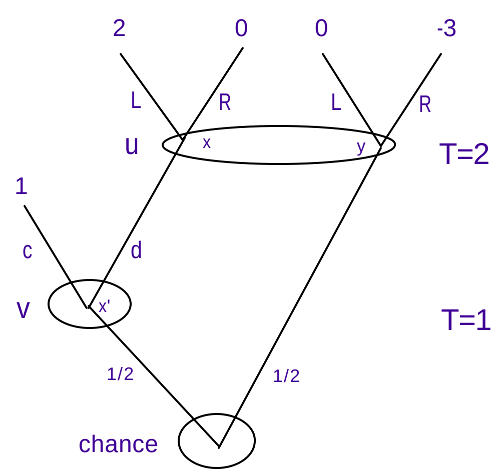

Figure 4

The one-player game of Figure 5 with the information partition U_1 = fu; v; wg satis…es a-loss recall.[4] However, Á is not acyclic, a fortiori, not a partial ordering in U_1.

[4] In fact, every one-player game without chance moves has a-loss recall, as long as Á is irre‡exive on U_1.

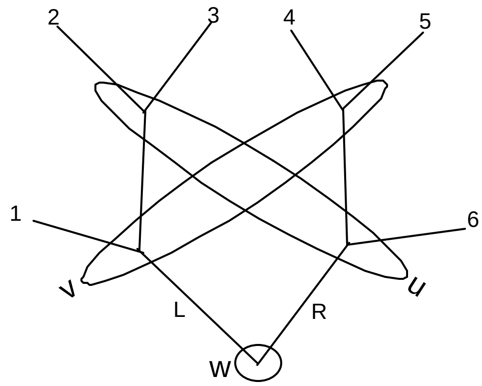

Figure 5

Perhaps all information partitions that satisfy both a-loss recall and the partial ordering property will satisfy occurrence memory. This also is not true. If we give the chance move to player 1 in the game of Figure 4, then the player will have a-loss recall (see footnote 4), but not occurrence memory. Nonetheless, $\prec$ is a partial ordering in U_1.

Finally, perhaps all information partitions that satisfy a-loss recall and acyclicity will give a partial ordering. This is also not the case. If we give the chance move in the game of Figure 3 to player 1, then the game satisfies a-loss recall and acyclicity, but $\prec$ is not a partial ordering in U_1 once again because of the lack of transitivity.

The point of giving such a variety of examples in this section is to emphasize that there is a large class of games of imperfect recall between absent-mindedness and occurrence memory. Many of those are personally time-structured and many others are not.

As a final note, we return to the alternative definition suggested in the Section 3.1 for the definition of $u \prec v$. The alternative was that $u \prec v$ iff for all $y \in v$, there exists a $x \in u$ such that $x \prec y$. In the game of Figure 4, information set v occurs at time $T = 1$ and u occurs at time $T = 2$. Thus we can say that v precedes u in time. However, since we cannot find $x' \in v$ with $x' \prec y$, we do not have $v \prec u$ by this definition. In this sense, this alternative definition does not capture our intentions.

4 Common Time-Structures

Often players need to think about what their opponents might have done in the past, or are likely to do in the future. This might be facilitated by having an ordering of a player's own decisions with those of his opponents. Such an ordering is possible if there is a common clock.

In Section 4.1, we define a common clock and give Theorem 4.1 that the acyclicty of $\prec$ in the set of all information sets in the game $U = \bigcup_{i \in N \cup \{0\}} U_i$

is equivalent to a game being commonly time-structured. This result is analogous to Theorem 3.1 obtained for personal clocks. In discussing the relationship between common clocks and memory, we find an interesting difference to our findings for personal clocks and memory.

The notion of a common clock brings up the possibility of different players moving at the same time. Simultaneous moves are, strictly speaking, not allowed in our current formulation of an extensive game. However, with some care, they can be included, and a variant of Theorem 4.1 is proved for games with simultaneous moves in Section 4.2.

4.1 Common Clocks

We extend the relation $\succ$ to the set of all information sets $U = \bigcup_{i \in N \cup \{0\}} U_i$ in the way analogous to how we extended $\succ$ to U_i. For $u, v \in U$, we write $u \succ v$ iff $x \succ y$ for some $x \in u$ and some $y \in v$.

Common Clock. A common clock T is a natural number-valued function on U such that for all $u, v \in U$,

$$u \succ v \text{ implies } T(u) < T(v). \tag{5.1}$$

A game Γ is called commonly time-structured when a common clock T can be found for Γ. Since (5.1) is required to hold everywhere in U, it follows that every commonly time-structured game is personally time-structured for each player. By paying attention to U rather than U_i we can prove the analogue of Theorem 3.1 for common clocks in the same manner as in the proof of Theorem 3.1.

Theorem 4.1 (Common Clock 1) Let Γ be an extensive game and let U denote the set of all information sets in the game. The following two statements are equivalent.
(1) The relation $\succ$ in U is acyclic.
(2) Γ is commonly time-structured.

A game that is personally time-structured for each player, may not be commonly time-structured. The game of Figure 1 is one such example. A common clock in this game would require associating a later time to Andy's information set than the time assigned to Tom's information set since $y \succ y'$. This is all fine until we realize that since $x' \succ x$, we must assign an earlier time to Tom's information set than the one assigned to Andy's. This impossibility was what led Tom into the back to the future cycle when he tried to put himself in Andy's shoes.

In spite of not being commonly time-structured, the game of Figure 1 happens to be a game of perfect recall. In fact, whenever a player i has only one information set, and $\succ$ is irreflexive in U_i, player i has perfect recall.

Our finding that a game may have perfect recall, but not be commonly time-structured, is in contrast to our finding that perfect recall (or even occurrence memory) implies the game is personally time-structured. An intuitive explanation for this difference is as follows. If a player has perfect recall, then he must be able to order his own past. If he could not order his own past, then he must have forgotten the order.[5] However, regarding the past moves of other players, no implication on ordering can be derived from a player having perfect recall. If the player never observed the moves of others, then he might not be able to order those moves with his own.

As far as implications of common time-structures for conditions on memory known in the literature, we find that absent-mindedness is ruled out in commonly time-structured games, as it was for personally time-structured games. However, all other known conditions on memory are independent of whether or not the game is commonly time-structured.

When a game happens to be commonly time-structured, the players and game theorists can reason more easily through it. A deterrent to studying games with imperfect recall has been the problem that without perfect recall, a decision maker might not be able to order his own past or future. In such a case, reasoning about the game might be difficult. However, it is difficult for much the same reasons that Tom found reasoning about the game of Figure 1, with perfect recall, to be difficult. In short, the difficulty in reasoning we are alluding to comes from the lack of an appropriate time-structure irrespective of whether or not the players have perfect recall.

In the Section 5 we show how common time-structures can be used to extend results on backward induction reasoning from games of perfect information to games with imperfect recall. This emphasizes that such results are related to the time structure properties embodied in perfect information games, but not to conditions on memory.

We conclude this section by discussing the relationship between our work and other notions of time in the game theory and the computer science systems literature.

Von Neumann and Morgenstern's [14] definition of an extensive game is based implicitly on the notion of a "common clock". The clock is used by them to restrict the allowed moves in the game. On page 49 of their book, they write: "The moves themselves we denote by $M_1, \ldots, M_\nu$, and we assume that this is the chronological order in which they are prescribed to take place."

Another place where time arises in the game theory literature is in the context of a repeated game. A repeated game involves the repetition of a "base" game over time. Time here is commonly agreed upon. Hence,

[5] We are requiring here that a player is conscious of his own movements at the time he is moving. To have forgotten something implies having known it at some time. If a person is not conscious of his movements, then he might never know them, and thus he might not be regarded as having forgotten them.

we have a common clock for the repetitions of the base game. However, this notion of time is a partial one since there is no restriction on the time structure for the base game.

Time also appears is in the computer science systems literature of Fagin, Halpern, Moses, and Vardi [2]. In this literature, time is explicitly modelled as part of a system. Their findings are applicable since they show that extensive games can be modelled as “multi-agent systems”. Their notion of a “shared synchronous clock” is rather close to our notion of a common clock.

In each of these cases cited, time appears either implicitly or explicitly in the form of a “common clock”. In our analysis, we ask whether or not a game is commonly time-structured or personally time-structured. The notion of a commonly time-structured game is based on the notion of a common clock, and thus clearly included to some extent in each of these other works. To my knowledge, however, analysis of “personal clocks” has not been discussed elsewhere in the game theory or systems literature.

4.2 Simultaneous moves

With a common clock, we might imagine that players can move simultaneously. Many games involve simultaneous moves, and it would be nice to be able to apply our result to those games.

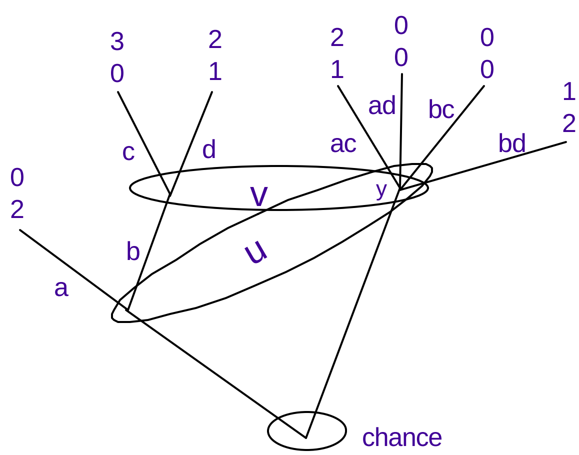

Figure 6

To include simultaneous moves, the definition of the player partition $P = \{P_0, P_1, \ldots, P_n\}$ is replaced by the sets $P_0, P_1, \ldots, P_n$, but now $P_i \cap P_j$ may be non-empty for $i \neq j$. Information sets of different players may also intersect, and thus the information pattern U may not be a partition. However, for each player i, U_i can again be constrained to be a partition of P_i. Condition (3.c) on the information partition of each player must be replaced by an alternative condition to ensure a player cannot distinguish between nodes in the same information set, and care needs to be taken to adjust the definitions of the choice partition. The interested reader is referred to Dubey and Kaneko [1] for details, but those details are not needed to follow the arguments given in the rest of this section.

The game of Figure 6 is a two-player game with $U_1 = \{u\}$ and $U_2 = \{v\}$. Since $y \in u$ and $y \in v$, the game involves simultaneous moves by the two players when y is reached. The choices of player 1 at u are a and b and the choices of player 2 at v are c and d. In the case that node y is reached, we get four alternatives, ac, ad, bc, and bd. At every other node, there are only two alternatives. This is the sense in which we must take care in defining the choice partition and (3.c) needs to be adjusted.

In our previous definition of a common clock given by (5.1), we required only that if one move occurs after another, then it should be given a later time. Now, because we allow the moves of several players to occur simultaneously, we should require that these moves are given the same time by our common clock.

A common clock in an extensive game with simultaneous moves is a natural number-valued function T on U such that for all $u, v \in U$, (5.1) and

$$u \cap v \neq \emptyset \text{ implies } T(u) = T(v). \tag{5.2}$$

An extensive game with simultaneous moves is called commonly time-structured when we can find a common clock satisfying (5.1) and (5.2).

The game of Figure 6 is one where a common clock satisfying both (5.1) and (5.2) cannot be found. The problem is that (5.1) requires $T(u) < T(v)$ since $u \prec v$, and (5.2) requires $T(u) = T(v)$ since $u \cap v \neq \emptyset$.

The limit for common clocks in games with simultaneous moves can also be defined in terms of acyclicity. The definition of acyclicity used here is based on the relation $\precsim$ obtained from $\prec$ by defining $u \precsim v$ iff $u \prec v$ or $u \cap v \neq \emptyset$. The relation $\precsim$ is acyclic in U iff for any finite sequence of elements $u_1, \ldots, u_k$ of U, if $u_1 \precsim u_2 \precsim \cdots \precsim u_k$ then $u_k \not\prec u_1$. We obtain the following theorem.

Theorem 4.2 (Common Clock 2) Let Γ be an extensive form game with simultaneous moves and let U denote the set of all information sets in the game. The following two statements are equivalent.
(1) The relation $\precsim$ in U is acyclic.
(2) Γ is commonly time-structured.

Proof: (1) implies (2). Suppose that $\precsim$ is acyclic in U. Let $u \in U$ be arbitrarily given. Define $I(u)$ as the set of information sets that have a non-empty intersection with u. A finite chain is a finite sequence of information sets $u_1, \ldots, u_k$ that satisfies $u_1 \precsim u_2 \precsim \cdots \precsim u_k$. The length of a chain is defined here as the number of instances of $\prec$ in the chain. Assign $T(u)$ equal to the maximal length of a chain $u_1, \ldots, u_k$ with the last element $u_k \in I(u)$.

Acyclicity implies that if $u_1 \precsim \cdots \precsim u_i \prec u_{i+1} \precsim \cdots \precsim u_k$, then $u_k \neq u_1$. Hence, since the game is finite and $\precsim$ is acyclic in U, $T(u)$ is a natural number. Since u was chosen arbitrarily, we can use the same procedure to assign, to each $v \in U$, a natural number $T_i(v)$.

Let $u; v \in U$ be arbitrarily given. If $u \prec v$, then (5:1) is proved in the same way as (3.1) in Theorem 3.1.

If $u \cap v \neq \emptyset$, then $v \in I(u)$ and $u \in I(v)$. Thus $T(u) = T(v)$ and (5.2) holds.

(2) implies (1). Suppose that $\precsim$ is not acyclic in U. Then there is a finite sequence of elements of U with $u_1 \precsim u_2 \precsim \cdots \precsim u_k$ and $u_k \prec u_1$. But then a common clock requires $T(u_1) < T(u_1)$, which is impossible. $\square$

If player 2's information set in Figure 6 is partitioned into two singleton information sets, then the game still involves simultaneous moves at y, but the relation $\precsim$ becomes acyclic in U. By Theorem 4.2, the game is now commonly time-structured.

5 Backward Induction

In games of perfect information, the backward induction method is often used to obtain a solution. This involves working back from the uppermost personal information sets of the tree to its root. In a commonly time-structured game, we can use a common clock and work back in time from the latest time moves of personal players in the game to their earliest moves. Hence, it might be possible to apply a backward induction algorithm to find equilibrium solutions to the game. In this section, we show that this can be done for a class of games that are called belief independent.

Let's return to the one-player game of Figure 4 which is a commonly time-structured game of imperfect recall. One common clock in this game assigns[6] $T(\text{chance}) = 0$, $T(v) = 1$, and $T(u) = 2$. We start at the latest time, $T = 2$, which brings us to information set u. Observe that what the player should do at u to maximize his payoff for the remainder of the game, does not depend on his beliefs. Regardless of the beliefs he assigns to nodes x and y, his optimal choice will be L at u. We fix this choice and move down to $T = 1$ where we find the personal information set v. Taking the choice of L at u as given, the optimal choice at v is d. There are no more personal information sets, so we now have a backward induction solution of d at v and L at u. In this game, the solution is unique, but in some games there will be multiple solutions.

Since this example is a one-player game, the induction required only a personal clock. However, to make things work in the multiple player case, we might need a common clock.

[6] We allow natural numbers to include 0.

If every information set of every player is a singleton, then the game is called one of perfect information. For a perfect information game, we have the result that every subgame perfect equilibrium[7] can be obtained by a backward induction. Furthermore, a pure strategy subgame perfect equilibrium exists in such a game. We want to extend these results to games with imperfect information and perhaps games with imperfect recall. To do this, we should use a solution concept like perfect equilibrium, or sequential equilibrium.

We chose the concept of sequential equilibrium here [7]. One reason is that perfect equilibrium sometimes refines the set of subgame perfect equilibrium even in a game of perfect information (Kline [5]). Hence, it is not true that backward induction in a game of perfect information always gives a perfect equilibrium. It is true, on the other hand, that backward induction in a game of perfect information always gives a strategy supported by a sequential equilibrium.

To define a sequential equilibrium, we need the notion of a system of beliefs and the notion of a behavior strategy. A system of beliefs is a function μ on X satisfying: (a) $\mu(x) \in [0,1]$ for all $x \in X$, and (b) $\sum_{x \in u} \mu(x) = 1$ for all $u \in U$. A behavior strategy of player i is a function b_i that assigns to each $u \in U_i$, a probability distribution b_{iu} over the set C_u of choices at u. Behavior strategies allow a player to randomize his choices each time his information set is reached.

We denote the set of behavior strategies of player i by B_i. Each b_{iu} is called a local strategy of player i at u. We denote the set of local strategies of player i at u by B_{iu}. We call b_{iu} a local pure strategy iff b_{iu} assigns probability 1 to some choice $c \in C_u$. A behavior strategy b_i is called a pure strategy iff b_i assigns a local pure strategy to each $u \in U_i$. An n-tuple $b = (b_1, \ldots, b_n)$ of behavior strategies, one for each player, is called a strategy combination. An ordered pair (b, μ) where b is a strategy combination and μ is a system of beliefs is called an assessment.

Given a strategy combination $b = (b_1, \ldots, b_n)$, a node $x \in X$, and a player $i \in N$, the expected payoff of player i conditional on being at x is defined by $H_{ix}(b) = \sum_{z \in Z_x} p(z \mid x, b) h_i(z)$ where $Z_x = \{z \in Z : x \prec z\}$ and $p(z \mid x, b)$ is the probability of reaching endnode z when we are currently at node x and b is being used in the continuation of the game. For an assessment (b, μ) and an information set u where a personal player i moves, the expected payoff of player i conditional on being at u is defined by $H_{iu}(b, \mu) = \sum_{x \in u} \mu(x) H_{ix}(b)$.

We will use (b_i^0, b_{-i}) to denote the strategy combination obtained from

[7] A subgame perfect equilibrium is a type of Nash equilibrium which is regarded as appropriate in games of perfect information. We define Nash equilibrium later in this section. Readers interested in the concept of subgame perfection are referred to Selten (1975).

b by replacing the behavior strategy b_i of player i by b_i'. We will also use (b_{iu}', b_{-iu}) to denote the strategy combination obtained from b by replacing the local strategy of the player i moving at u by b_{iu}'.

We say that an assessment (b, μ) is sequentially rational at information set u of personal player i iff $H_{iu}(b, \mu) \geq H_{iu}((b_i', b_{-i}), \mu)$ for all $b_i' \in B_i$. An assessment (b, μ) is called sequentially rational iff (b, μ) is sequentially rational at each personal information set in U. An assessment (b, μ) is consistent iff there is a sequence of completely mixed[8] strategy combinations $\{b^k\}_{k=1}^{\infty}$ satisfying both $\lim_{k \to \infty} b^k = b$, and for each $u \in U$ and each $x \in u$, $\mu(x) = \lim_{k \to \infty} \frac{p(x, b^k)}{\sum_{y \in u} p(y, b^k)}$. We can now define a sequential equilibrium.

An assessment (b, μ) is a sequential equilibrium iff (b, μ) is sequentially rational and consistent.

The class of games we will extend backward induction arguments to is those games where expected payoff maximization at every information set is independent of beliefs there.

Belief independent games: A game Γ is belief independent iff for any two systems of beliefs μ and μ' and all $i \in N$, if $u \in U_i$, $b_{-i} \in B_{-i}$, and b_i^* maximizes $H_{iu}((\cdot, b_{-i}), \mu)$ over B_i, then b_i^* maximizes $H_{iu}((\cdot, b_{-i}), \mu')$ over B_i.

This definition includes perfect information games. It also includes some games with imperfect recall like the game of Figure 4.

The notion of belief independence used here should not be confused with a result due to Selten ([13], Lemma 4) that for a game of perfect recall, the conditional probability of being at a node x given we are at the information set u containing x is independent of the strategy of the player moving at u. Because such conditional probabilities are often called beliefs of the player, Selten's result can be interpreted as a type of belief independence.

It is easily distinguished from the usage in this paper by observing that Selten's belief independence means the beliefs at an information set are independent of the strategy chosen by the player moving there. In this paper, belief independence means the strategy chosen by the player to maximize his conditional payoff at an information set is independent of his beliefs there.

Kaneko and Kline ([3], Lemma 2.6) showed that Selten's belief independence is equivalent to perfect recall. From that result, and the well know result that games with perfect recall have a Nash equilibrium in behavior strategies, we obtain the result that all games with Selten's belief independence have a Nash equilibrium in behavior strategies. As we shall soon see,

[8] A strategy combination is called completely mixed if it assigns a strictly positive probability to each choice at each information set in the game. Since chance assigns a strictly positive probability to each choice at each $u \in U_0$, it follows that if b is a completely mixed strategy combination, then $p(x, b) > 0$ for all $x \in K$.

the notion of belief independence used in the current paper also guarantees a Nash equilibrium in behavior strategies.

Consider now the following backward induction algorithm to compute strategy combinations in a game.

Backward Induction Algorithm (BIA):

(1) Start with an arbitrary assessment $(b;\mu)$ for Γ, and a common clock T defined on U. Let t_m denote the time of the last move in the game, i.e., $T(u) = t_m$ for some $u \in U$ and $T(v) \le t_m$ for all $v \in U$.

(2) At each personal information set u such that $T(u) = t_m$, replace the local strategy b_{iu} in b by a local strategy b^*_{iu} that satisfies $H_{iu}((b^*_{iu}; b_{-iu}); \mu) \ge H_{iu}((b_{iu}; b_{-iu}); \mu)$ for all $b_{iu} \in B_{iu}$. The new strategy combination obtained is used in the continuation of this procedure.

(3) Repeat (2) inductively, moving back in time from t_m, to $t_m - 1$, down to the earliest time in the game. If at some step (2) there were multiple solutions b^*_{iu}, then repeat the procedure for each possible maximizing b^*_{iu} until all possibilities are exhausted. In this case multiple strategy combinations will be obtained by the algorithm.

Because the system of beliefs μ inputted into the algorithm is arbitrary, μ might not form a sequential equilibrium with any strategy combination b^* obtained by the algorithm BIA. We say that a strategy combination b^* supports a sequential equilibrium iff there is a system of beliefs μ such that the assessment $(b^*;\mu)$ is a sequential equilibrium.

We will also give one result that has to do with the notion of a Nash equilibrium [9] which is based on the ex ante expected payoff of each player. For a strategy combination b, the ex ante expected payoff of player i is defined by $H_i(b) = \sum_{z \in Z} p(z; b)h_i(z)$, where $p(z; b)$ denotes the probability of reaching node z when b is used in the game.

A strategy combination b^* is a Nash equilibrium iff for all $i \in N$, $H_i(b) \ge H_i(b'_i; b_{-i})$ for all $b'_i \in B_i$.

We can now give the main result of this section. A proof is given at the end of this section.

Theorem 5.1 Let Γ be a commonly time-structured game that is belief independent.
(a) For each starting assessment $(b;\mu)$, every strategy combination b^* obtained by BIA supports a sequential equilibrium.
(b) If the assessment $(b^*;\mu)$ is a sequential equilibrium, then b^* is obtained by BIA.
(c) If the assessment $(b^*;\mu)$ is a sequential equilibrium, then b^* is a Nash equilibrium.
(d) A sequential equilibrium exists in which each player uses a pure strategy.

Some comments on Theorem 5.1 are in order. First, one might wonder why we included in (c) the fact that each b^* is a Nash equilibrium. Aren't all sequential equilibria also Nash equilibria? The answer is yes for games of perfect recall, but no for some games of imperfect recall, even ones that are commonly time-structured. This is shown in Kline [6]. Thus a benefit of Theorem 5.1 (c) is that for the class of commonly time-structured and belief independent games, every sequential equilibrium is a Nash equilibrium.

The combinations of (a) and (b) imply that the set of strategy combinations obtained by BIA is precisely the set of strategy combinations supported by sequential equilibria. Thus, we find that the concept of sequential equilibrium captures backward induction reasoning in belief independent games.

Regarding (d), we have existence problems for behavior strategies in games with imperfect recall. So this part gives an existence result for a class of games with imperfect recall.

Before proving Theorem 5.1, we discuss what might happen if we try to apply the backward induction algorithm BIA to a game that is not belief independent or not commonly time-structured.

Suppose first that the game is not belief independent, but it happens to be commonly time-structured. In this case, we will still get at least one behavior strategy combination b^* from BIA for each starting assessment $(b; \mu)$. However, the set of strategy combinations obtained by BIA may depend on the system of beliefs μ inputted to the algorithm. Furthermore, a strategy combination b^* obtained by BIA may not be supported by a sequential equilibrium. Lastly, some behavior strategies that are supported by a sequential equilibrium may not be obtained by BIA.

One might try to weaken belief independence a bit and still obtain all the results of Theorem 5.1. One natural weakening would be to require that the system of beliefs μ used in BIA be such that there is a strategy combination b^0 such that $(b^0; \mu)$ is a consistent assessment. The set of all such systems of beliefs could be regarded as the set of consistent beliefs. We could then define a game as belief independent for consistent beliefs iff payoff maximization at each information set is independent of consistent beliefs. Unfortunately, now the solution to BIA may not be a sequential equilibrium.

For example, consider the game obtained from Figure 4 by changing only the furthest right payoff from -3 to 3. That game is no longer belief independent in the general sense since the optimal local choice at u is $b_{1u}(R) = 1$ if $\mu(y) > \frac{2}{5}$, but it is $b_{1u}(R) = 0$ if $\mu(y) < \frac{2}{5}$. If we restrict attention only to consistent beliefs, then $\mu(y) \geq \frac{1}{2}$. Over this set of beliefs, the game is belief independent. The BIA based on any system of consistent beliefs always gives the unique solution $b^*_{1u}(R) = 1$ and $b^*_{1v}(c) = 1$. This is not supported by a sequential equilibrium, however, since sequential rationality at v requires $b_{1v}(c) = 0$ and $b_{1u}(R) = 0$ for any system of beliefs.[9] This example shows that the concept of sequential equilibrium does not capture backward induction in some games of imperfect recall.

Now, suppose a game is not commonly time-structured. In this situation we will not be able to apply BIA since we cannot find the latest time information sets. For example, in the game of Figure 1 we cannot put Andy's information set as latest, since it precedes Tom's. Similarly, Tom's cannot be put latest, since it precedes Andy's.

So how might we proceed in this game? Since this game has perfect recall, it is well known that it has a Nash equilibrium in behavior strategies for each assignment of chance probabilities p.

There are two issues. One is how to assign the chance probabilities. Since they affect the decisions of both players, we would need to obtain agreement on those probabilities.

Once the chance probabilities are agreed upon, and assigned, we could simply look for a Nash equilibrium. This brings us to the second issue. While this type of reasoning breaks through the potential back to the future problem, it does involve a quite sophisticated type of analysis. It is not clear to me how the players in the game of Figure 1 might come to that equilibrium without some game theorist suggesting it to them. Backward induction reasoning, on the other hand, is much simpler to apply to in extensive game.

The following two lemmas are used in the proof of Theorem 5.1. Lemma 5.2 is given without proof since it is commonly known in the equivalent form that each player can do as well against the strategy combination of his opponents by using a pure strategy.[10] Lemma 5.3 characterizes belief independence.

Lemma 5.2 Let Γ be a game that is personally time-structured for player i and let μ be a belief system. For each information set $u \in U_i$ and each $b_{-iu} \in B_{-iu}$, $H_{iu}((\cdot; b_{-iu}); \mu)$ is maximized over B_{iu} by a local pure strategy.

Lemma 5.3 The following two statements are equivalent.
(a) Γ is a belief independent game.
(b) For each $i \in N$, each $u \in U_i$, each $b_{-i} \in B_{-i}$, and each system of beliefs μ, if b_i^* maximizes $H_{iu}((\cdot; b_{-i}); \mu)$ then b_i^* maximizes $H_{ix}(\cdot; b_{-i})$ over B_i at each $x \in u$.

Proof: (a) implies (b). Let Γ be belief independent and let μ be a system of beliefs. By belief independence of Γ, if b_i^* maximizes $H_{iu}((\cdot; b_{-i}); \mu)$ over B_i, then b_i^* maximizes $H_{iu}((\cdot; b_{-i}); \mu')$ for any other belief system μ'. In particular, we can successively choose belief systems that assign probability

[9] It is, however, the unique Nash equilibrium. See Kline [6] for results comparing Nash equilibrium and sequential equilibrium in games of imperfect recall.

[10] Lemma 5.2 can be proved for every game where U_i satisfies conscious-mindedness.

1 to each node $x \in u$ to obtain the result that b_i^* maximizes $H_{ix}(\cdot; b_{-i})$ over B_i at each $x \in u$.

(b) implies (a). Let μ be any system of beliefs. By (b), if b_i^* maximizes $H_{iu}((\cdot; b_{-i}); \mu)$, then b_i^* maximizes $H_{ix}(\cdot; b_{-i})$ at each $x \in u$. Suppose b_i^* is such a maximizer. Since $H_{iu}(\cdot; b_{-i}); \mu) = \sum_{x \in u} \mu(x) H_{ix}(\cdot; b_{-i})$, i.e., it is just an average of the $H_{ix}(\cdot; b_{-i})$'s over $x \in u$, we have that b_i^* maximizes $H_{iu}(\cdot; b_{-i}); \mu')$ for any other system of beliefs μ'. $\square$

Proof of Theorem 5.1: Suppose Γ is a commonly time-structured and belief independent finite game. Then we can assign a common clock T as required in step (1) of BIA. We fix a common clock T for Γ in what follows.

(a) Suppose b^* is obtained by BIA under belief system μ'. We need to show that b^* and μ is a sequential equilibrium for some belief system μ.

Select a sequence of completely mixed behavior strategy combinations $\{b^k\}_{k=1}^{\infty}$ that converges to b^*. Such a sequence can always be selected. For example, if $b_u^*(c) < 1$, then choose $b_{iu}^k(c) = b_{iu}^*(c) + \frac{1}{k}$. If $b_{iu}^k(c) = 1$, then choose $b_{iu}^k(c) = b_{iu}^*(c) - \frac{1}{k}$. For large enough k, these define completely mixed behavior strategies that converge to b^* as $k \to \infty$. Let μ be the associated system of beliefs. Then b^* and μ are consistent.

We prove sequential rationality of the assessment $(b^*; \mu)$ by induction over the personal information sets of U. Let $P(u)$ be the property that $(b^*; \mu)$ is sequentially rational at personal informations set u. A personal information set u is called uppermost iff there is no personal information set $v \in U$ such that $u \succ v$. The basis for induction is: (i) $P(u)$ holds for all uppermost information sets u. The inductive step is: (ii) Let u be an arbitrary, but not uppermost, information set. Suppose $P(v)$ for all v such that $T(v) > T(u)$. Then $P(u)$.

First we prove the basis (i). Let u be an arbitrary uppermost information set of a personal player. Since b^* is chosen by BIA given the initial assessment $(b; \mu')$, the local strategy b_{iu}^* satisfies:

$$H_{iu}((b_{iu}^*; b_{-iu}); \mu') \geq H_{iu}((b_{iu}'; b_{-iu}); \mu') \text{ for all } b_{iu}' \in B_{iu}. \tag{5.1}$$

Since u is uppermost, $H_{iu}((\cdot; b_{-iu}); \mu')$ is independent of b_{-iu}. Hence, by (5.1) we obtain:

$$H_{iu}(b^*; \mu') \geq H_{iu}((b_i'; b_{-i}^*); \mu') \text{for all } b_i' \in B_i. \tag{5.2}$$

Since the game is belief independent, we can change the beliefs from μ' to μ in (5.2) to obtain:

$$H_{iu}(b^*; \mu) \geq H_{iu}((b_i'; b_{-i}^*); \mu) \text{ for all } b_i' \in B_i. \tag{5.3}$$

Next we prove the induction step (ii). Let u be an arbitrary, but not uppermost, information set of a personal player i. Suppose $P(v)$ for all personal information sets v such that $T(v) > T(u)$. For each $x \in u$, define $IS(x) = \{y \in P_i : x \succ y$ and there is no $y' \in P_i$ with $x \succ y' \succ y\}$, that is, $IS(x)$ is the set of nodes of player i that are immediate successors of x when we restrict the binary relation $\succ$ to P_i. We can partition the endnodes Z_x into the set of nodes that succeed some $y \in IS(x)$, which we call $Z_{IS(x)}$, and the set of nodes that do not, $(Z_x - Z_{IS(x)})$. We can write the conditional expected payoff of player i from being at u when any strategy $b_i \in B_i$ is chosen against b^*_{-i} and beliefs are μ' as:

$$H_{iu}((b_i; b^*_{-i}); \mu') = \sum_{x \in u} \mu'(x) A[x; (b_i; b^*_{-i})], \tag{5.4}$$

where:

$$A[x; (b_i; b^*_{-i})] = \sum_{y \in IS(x)} p(y \mid x; (b_i; b^*_{-i})) H_{iy}(b_i; b^*_{-i}) +$$

$$\sum_{z \in Z_x - Z_{IS(x)}} p(z \mid x; (b_i; b^*_{-i})) h_i(z). \tag{5.5}$$

Observe that for each y in (5.5), the maximizing choice of b_i for $H_{iy}(b_i; b^*_{-i})$ does not depend on b_{iu}, since $x \succ y$ and the game is commonly time-structured. Also, the inductive hypothesis applies to the information set v containing y. Hence, by Lemma 5.3, $H_{iy}(b^*) \geq H_{iy}(b'_i; b^*_{-i})$ for all $b'_i \in B_i$.

Observe next that since each y in (5.5) is chosen as an immediate successor of x when we restrict attention to nodes in P_i, the term $p(y \mid x; (b'_i; b^*_{-i}))$ does not depend on any part of b_i except possibly b_{iu}. By similar reasoning, for each $z \in (Z_x - Z_{IS(x)})$, the term $p(z \mid x; (b_i; b^*_{-i}))$ does not depend on any part of b_i except possibly b_{iu}. Let $b_i \in B_i$ and $b_{iu} \in B_{iu}$. We use $(b'_{iu}; b_{i-u})$ to denote the behavior strategy that chooses b'_{iu} at u, and according to b_i elsewhere in U_i.

By the arguments made in the previous two paragraphs about nodes y in (5.5), we have that for each $b_{iu} \in B_{iu}$,

$$H_{iu}(((b_{iu}; b^*_{i-u}); b^*_{-i}); \mu') \geq H_{iu}(((b_{iu}; b'_{i-u}); b^*_{-i}); \mu') \text{ for all } b'_i \in B_i. \tag{5.6}$$

Since b^* is obtained by the BIA,

$$H_{iu}(((b^*_{iu}; b^*_{i-u}); b^*_{-i}); \mu') \geq H_{iu}(((b_{iu}; b^*_{i-u}); b^*_{-i}); \mu') \text{ for all } b_{iu} \in B_{iu}. \tag{5.7}$$

The combination of (5.6) and (5.7) implies:

$$H_{iu}((b^*_i; b^*_{-i}); \mu') \geq H_{iu}((b'_i; b^*_{-i}); \mu') \text{ for all } b' \in B_i. \tag{5.8}$$

Hence, $(b^*; \mu')$ is sequentially rational at u. Since the game is belief independent, $(b^*; \mu)$ is also sequentially rational at u, i.e., P(u).

(b) Let $(b^*; \mu)$ be a sequential equilibrium. We prove, by induction on the personal information sets in U, that the BIA obtains b^* given any initial assessment $(b; \mu')$. We fix $(b^*; \mu)$ and $(b; \mu')$ in what follows. Let u be an arbitrary information set of personal player i. Since b^* and μ is a sequential equilibrium,

$$H_{iu}(b^*; \mu) \geq H_{iu}((b'_{iu}; b^*_{-iu}); \mu) \text{ for all } b'_{iu} \in B_{iu}: \tag{5.9}$$

By belief independence, we can replace μ in (5.9) by μ' to obtain:

$$H_{iu}(b^*; \mu') \geq H_{iu}((b'_{iu}; b^*_{-iu}); \mu') \text{ for all } b'_{iu} \in B_{iu}. \tag{5.10}$$

We will use (5.10) in our induction proof.

Let Q(u) be the property that the local strategy b^*_{iu} at personal information set u is chosen by BIA. The basis for induction is: (i) Q(u) at each uppermost personal information set u. The inductive step is: (ii) Let u be an arbitrary, but not uppermost, personal information set. Suppose Q(v) for all personal information sets v such that $T(v) > T(u)$. Then Q(u).

First, we prove the basis (i). Suppose u is an arbitrary uppermost personal information set of player i. Then $H_{iu}(\cdot; \mu')$ is independent of the local strategies of personal players other than perhaps the choice of i at u. Hence, we can replace b^*_{-iu} in the left and right hand sides of (5.10) by b_{-iu} to obtain:

$$H_{iu}((b^*_{iu}; b_{-iu}); \mu') \geq H_{iu}((b'_{iu}; b_{-iu}); \mu') \text{ for all } b'_{iu} \in B_{iu}. \tag{5.11}$$

But then b^*_{iu} is chosen by the BIA, i.e., Q(u).

Next, we prove the inductive step (ii). Suppose u is and arbitrary, but not uppermost, personal information set of player i. Suppose that Q(v) for all personal information sets v such that $T(v) > T(u)$. Let $U(u+) = \{v : T(v) > T(u)\}$. Let b'' denote the strategy combination obtained from b by replacing the local strategy b_{iv} by b^*_{iv} at each $v \in U(u+)$. Notice that this is precisely the strategy that will be given by BIA when we reach u. Since $H_{iu}(\cdot; \mu')$ in (5.10) is independent of personal local strategies other than perhaps those in $U(u+)$ or at u; we obtain from (5.10) that:

$$H_{iu}((b^*_{iu}; b''_{-iu}); \mu') \geq H_{iu}((b'_{iu}; b''_{-iu}); \mu') \text{ for all } b'_{iu} \in B_{iu} \tag{5.12}$$

But then b^*_{iu} is chosen by the BIA, i.e., Q(u).

(c) Suppose that $(b^*; \mu)$ is a sequential equilibrium in Γ. Consider an arbitrary personal player i. Let $u_1; ....; u_k$ denote the information sets of player i, that is, $U_i = \{u_1; ....; u_k\}$. We can use U_i to partition the endnodes into the sets $Z_0; Z_1; ....; Z_k$. The set Z_0 consists of all endnodes that are not successors of any $u \in U_i$. The nodes in Z_1 are the endnodes that are successors of u_1 and without an earlier predecessor in U_i, that is $Z_1 = \{z \in Z : u_1 \prec z$ and if $v \in U_i$ and $v \prec u_1$ then $v \not\prec z\}$. We denote the set of nodes in u_1 with successors in Z_1 as $u_1(Z_1)$. The nodes of $Z_2, \ldots, Z_k$ and the sets $u_2(Z_2), \ldots, u_k(Z_k)$ are defined in the analogous way using $u_2, \ldots, u_k$.

For example, in the one-player game of Figure 4, we can let $u_1 = v$ and $u_2 = u$, so that $U_1 = \{u_1, u_2\}$ as required. Let's label the five endnodes from left to right as z_1 to z_5. Then Z_0 is empty, $Z_1 = \{z_1, z_2, z_3\}$, and $Z_2 = \{z_4, z_5\}$. Furthermore, $u_1(Z_1) = \{x'\}$ and $u_2(Z_2) = \{y\}$.

The ex ante expected payoff of any player i for the strategy combination (b_i, b^*_{-i}), where b_i is any strategy in B_i, can be written as:

$$H_i(b_i, b^*_{-i}) = \sum_{z \in Z_0} p(z; (b_i, b^*_{-i})) h_i(z) + \sum_{j=1}^{k} \sum_{y \in u_j(Z_j)} p(y; (b_i, b^*_{-i})) H_{iy}(b_i, b^*_{-i}). \qquad (5.13)$$

Observe first that for each $z \in Z_0$ the term $p(z; (b_i, b^*_{-i})$ is independent of b_i. Observe next that for each y in each $u_j(Z_j)$, sequential rationality of (b^*, μ) and belief independence implies by Lemma 5.3 that $H_{iy}(\cdot, b^*_{-i})$ is maximized over B_i by b^*_i. Furthermore, since each such y was chosen to have no earlier predecessor nodes of player i, we have that $p(y; (b_i, b^*_{-i})$ is independent of b_i. Consequently, b^*_i maximizes $H_i(b_i, b^*_{-i})$ over B_i.

Since the player i was chosen arbitrarily, we can apply this argument to each player $i \in N$ to obtain that b^* is a Nash equilibrium.

(d) Suppose that (b, μ^0) is an arbitrary starting assessment. Applying Lemma 5.2 over personal information sets we obtain a solution b^* to BIA with b^*_i is a pure strategy for each personal player i. By part (a) of the current theorem, b^* forms a sequential equilibrium with some belief system μ. $\square$

6 Conclusions

We introduced the notions of personal time-structures and common time-structures to extensive games. These structures help players and game theorists reason through a game. Personally time-structured games are ones in which each player can order his own moves. Commonly time-structured games are games in which all players agree on the ordering of all moves of all players.

We characterized a game as being personally time-structured for player i if and only if the relation $\prec$ describing precedence over information sets is acyclic in player i's information partition U_i. In much the same way, we characterized a game as being commonly time-structured if and only if the same relation Á is acyclic in the set of all information sets U of all players.

Personal time-structures were found to be implied by perfect recall, and by a condition of memory weaker than perfect recall known as occurrence memory. Common time-structures, however, are not implied by occurrence memory or even by perfect recall.

The condition of a game having a personal time-structure or common time-structure is a rather natural one in many economic and game theoretic settings. Extensive games with time structures allow us to explore many examples, including a large portion of games with imperfect recall, though they exclude absent-mindedness.

By focusing on commonly time-structured games, we found that backward induction arguments can be extended from games of perfect information to some games with imperfect recall. We showed that backward induction reasoning is captured by the notion of a sequential equilibrium in a class of games which we called belief independent games. This class of games includes some region of imperfect recall. However, we found by a simple example of a game that is not belief independent, that the notion of sequential equilibrium may be at odds with backward induction reasoning in a game of imperfect recall.

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