3-21-2001

Minimum memory for equivalence between ex-ante optimality and time-consistency

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March 21, 2001

Abstract

We give a necessary and sufficient condition on memory of a player in an extensive game for equivalence between *ex ante* optimality and time-consistency (for all payoff assignments). The condition is called *A-loss recall* and requires that each loss of a player's memory can be traced back to some loss of memory of his own action. A-loss recall is also shown to be a necessary and sufficient condition for the existence of a time-consistent strategy (for all payoff assignments) if the player is conscious-minded. *Journal of Economic Literature* Classification Numbers: C72, D80

1 Introduction

Although an extensive game has an explicit move order structure for decision making, the timing of when exactly the strategy choice is made has been treated in a somewhat ambiguous manner in the game theory literature. The classical *ex ante* optimality view (hereafter EA-optimality) asserts that the player constructs his strategy before the actual play of the game. In illustrative terms, he ties the hands of his future selves by his *ex ante* strategy choice. The time-consistency view takes the alternative position that the player cannot tie the hands of his future selves. Each time he is given the chance to move he will usurp control from his former self and decide whether or not to change his strategy for the continuation of the game. For games of perfect recall, the EA-optimal and time-consistency views actually lead to equivalent behavior by an expected utility maximizing player. This is a nice finding because it allows us to jump back and forth between the two interpretations at our convenience. In this sense, the timing of the strategy choice becomes irrelevant.

The potential distinction between EA-optimality and time-consistency was uncovered by Piccione and Rubinstein (1997) in a game involving an absent-minded driver. They found the seemingly paradoxical result that no EA-optimal

*The author thanks Mamoru Kaneko, participants at the Economic Theory Workshop at Flinders University and an anonymous referee for helpful comments.

strategy was time-consistent in that game. This finding has become known as the absent-minded driver's paradox. It sparked a new interest in games without perfect recall. An entire issue of *Games and Economic Behavior* (Issue 20, 1997) was devoted to the absent-minded driver's paradox and games without perfect recall. The papers in that issue discussed the nature of the paradox and how it might be resolved. Alternative definitions of time-consistency were suggested by Aumman, Hart and Perry (1997), Halpern (1997), Halpern and Groves (1997), and Battigalli (1997). Gilboa (1997) and Halpern (1997) suggested alternative formulations of the problem. Nevertheless, no consensus was reached.

In this paper, instead of focusing on how to resolve the paradox, we take an alternative research strategy of trying to find exactly when the divergence between time-consistency and EA-optimality emerges. This will hopefully give us a better understanding of the nature of the paradox. Our research yields a condition on memory that is weaker than perfect recall, but stronger than absent-mindedness. The condition is known as A-loss recall and was given by Kaneko and Kline (1995). A-loss recall requires that any forgetfulness of the player can be traced back to some loss of memory of his own actions. Nonetheless, a player with A-loss recall may forget what he learned as well as what he did.

We show that A-loss recall is sufficient for equivalence between EA-optimality and time-consistency. Conversely, A-loss recall is necessary for equivalence in the sense that whenever it is violated, there is some payoff assignment for which the equivalence breaks down. This result states that it is the lack of A-loss recall, not absent-mindedness, that generates the absentminded driver's paradox.

Kaneko and Kline (1995) found the condition of A-loss recall while exploring the potential of strategies to compensate for a player's forgetfulness. In their analysis, they made no reference to a solution concept, while the present paper explicitly compares two solution concepts: EA-optimality and time-consistency. In spite of this difference, Kaneko and Kline (1995) found that precisely the same condition, A-loss recall, is necessary and sufficient for mixed strategies to fully compensate for the player's forgetfulness. In Section 5 we connect the results of Kaneko and Kline (1995) to those of this paper by providing an intuitive explanation of why A-loss recall suffices for equivalence of the two solution concepts.

In the present paper we also consider the question of existence of a time-consistent strategy raised by Battigalli (1997, Figure 1). An EA-optimal strategy always exists in a one player game regardless of the information structure. We find that the existence of a time-consistent strategy is related to the condition of A-loss recall. When a player has A-loss recall, a time-consistent strategy exists by our equivalence result. The non-existence of a time-consistent strategy comes only when A-loss recall is violated. By focusing on conscious-minded players (those who are not absent-minded), we are able to show that whenever A-loss recall is violated, there is a payoff assignment for which no time-consistent strategy exists. In this sense, A-loss recall is found to also be necessary and sufficient for the existence of a time-consistent strategy. A player is regarded as being absent-minded by Piccione and Rubinstein (1997) if he arrives at the same information set more than once. Absent-mindedness and its relationship to A-loss recall is discussed in Section 4.

In summary, both the non-equivalence between time-consistency and EA-optimality and the non-existence of a time-consistent strategy occur only when the information structure of a game violates the condition of A-loss recall. If a player's imperfect recall can be traced back to some imperfect recall about his own actions, then neither problem arises. These results help us to better understand the nature of the absent-minded driver's paradox by highlighting exactly what kind of memory loss is required to generate the paradox.

The remainder of the paper is organized as follows. In Section 2 we give basic definitions. In Section 3 we show that A-loss recall is necessary and sufficient for the equivalence between EA-optimality and time-consistency for all payoff assignments. In Section 4 we discuss existence and in Section 5 we give an intuitive explanation of Theorem 1. Proofs are given in Section 6 and concluding remarks in Section 7.

2 Basic Definitions

2.1 Game Trees, Strategies, and A-loss Recall

We follow Selten (1975) for the description of a finite game tree and focus on one player games which are sometimes referred to as decision problems. A finite one person game is a septuple $\Gamma = (K, P, U, C, Z, \rho, h)$. K is a finite tree with the unique root node s where every play of the game starts. P denotes the set of nodes in the tree where the decision maker moves, and U partitions those nodes into information sets of the player with the restriction that nodes in the same information set must have the same number of choices. For an information set $u \in U$ we let A_u denote the finite set of choices available at u. We assume that A_u contains at least two choices for every $u \in U$. C denotes the set of nodes where the chance player (nature) moves. For a chance node $x \in C$ let A_x denote the finite set of choices there. We assume that A_x contains at least two choices for every $x \in C$. ρ is a completely mixed probability assignment over the choices at chance nodes, that is, at each chance node $x \in C$, ρ assigns a non-zero probability $\rho_x(c)$ to each choice $c \in A_x$. Z denotes the set of endnodes and h is a payoff assignment which assigns a real number $h(z)$ to each endnode $z \in Z$. An example of a game is given in Figure 1 below where the player's information partition $U = \{u, v, w\}$. The game starts with a chance move.

For some results in this paper we use a game structure $\Gamma(\cdot) = (K, P, U, C, Z, \rho, \cdot)$ defined excluding the payoff assignment h. For each payoff assignment h we obtain a game $\Gamma(h) = (K, P, U, C, Z, \rho, h)$.

The player's behavior in a game is described by a behavior strategy.

Behavior Strategy: A *behavior strategy* b is a function which assigns to each $u \in U$ a probability distribution b_u over the set A_u. We denote the set of behavior strategies by B.

In the game of Figure 1, one behavior strategy b is defined by $b_u(e) = 1/3$, $b_v(d) = 0$ and $b_w(a) = 1$. This behavior strategy instructs the player to randomize at u by choosing e with probability 1/3 and f with probability 2/3. At information set v it instructs him to choose c for certain and at w to choose a for certain.

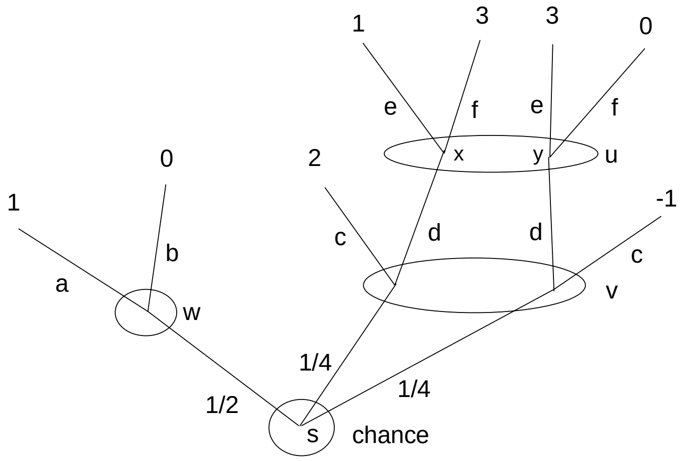

Figure 1

The successor relations $\succ_c$ and $\succ$ will be useful for both constructing conditions on memory and proving results. For nodes $x, y \in P \cup C \cup Z$ we write $x \succ_c y$ iff x can be reached from y via choice c at y. We write $x \succ y$ iff $x \succ_c y$ for some c at y. For node x and information set v we write $x \succ v$ iff $x \succ y$ for some node $y \in v$. Finally, for information sets $u, v \in U$ we write $u \succ v$ iff $x \succ v$ for some $x \in u$. We use $x \not\succ y$ and $x \not\succ_c y$ to denote the negation of $x \succ y$ and $x \succ_c y$ respectively.

The perfect recall condition requires a person to recall both what he did and learned. In terms of the successor relations it can be defined as follows.

Perfect Recall: The player has *perfect recall* iff for any $u, v \in U$, $x, y \in u$, and $c \in A_v$, $x \succ_c v$ implies $y \succ_c v$.

The game of Figure 1 is an example of a game with perfect recall. We now give a weakening of perfect recall which will prove to be of great import to our analysis. It was introduced by Kaneko and Kline (1995).

A-loss Recall: The player has *A-loss recall* iff for any $u, v \in U$, $x, y \in u$, and $c \in A_v$, $x \succ_c v$ implies (1) $y \succ_c v$ or (2) there is a $w \in U$ such that $x \succ_d w$ and $y \succ_e w$ for distinct $d, e \in A_w$.

Notice that the requirement that (1) be satisfied is just that the player has perfect recall. Hence, all players with perfect recall have A-loss recall. However, A-loss recall allows a player to forget things he learned and did. When he forgets something, another condition must be met, namely condition (2). Condition (2) can be interpreted as requiring that each loss of memory can be traced back to some loss of memory of the player's own previous actions.

The player in the game of Figure 2 has the information partition $U = \{v1, v2, u, w\}$. The player does not have perfect recall since at u he will have forgotten what he did at w and what he learned at $v1$ or $v2$ about the chance move. Nevertheless, the player has A-loss recall. Consequently, we can trace back his loss of memory at u about the move of chance to some loss of memory about his previous action at w. For example, if he knew that he chose L at w, then he would know at u that the chance move must have been f. If the chance move had been e, then he would never have reached u after having chosen L at w.

The player in the game of Figure 3 with the information partition $U = \{w, u, v\}$ does not have A-loss recall. To see this, notice that $x, y \in u$ and $x \succ_c v$ but $y \not\succ_c v$ so condition (1) of A-loss recall is not satisfied. Condition (2) of A-loss recall is not satisfied either since there is no information set $w \in U$ with both $x \succ w$ and $y \succ w$.

It is left to the reader to check that the player in the game of Figure 4 has A-loss recall and the player in the game of Figure 5 does not.

A convenient property of a player with A-loss recall is that each information set $u \in U$ has a unique "earliest-complete-own-predecessor" $v \in U$.

Take an information set $u \in U$ as given and define the sets of his own preceding and following information sets for u as $U_p = \{v \in U : u \succ v\}$ and $U_f = \{v \in U : v \succ u\}$.

We say that an information set $v \in U_p \cup \{u\}$ is an *earliest-complete-own-predecessor of* u iff (1) $V_p = \emptyset$, and (2) $v \neq u$ implies $x \succ v$ for all $x \in u$.

The earliest-complete-own-predecessor of an information set u may be u itself. In the game of Figure 2, w is the earliest-complete-own-predecessor of w. In fact, w is the earliest-complete-own-predecessor of every information set in U.

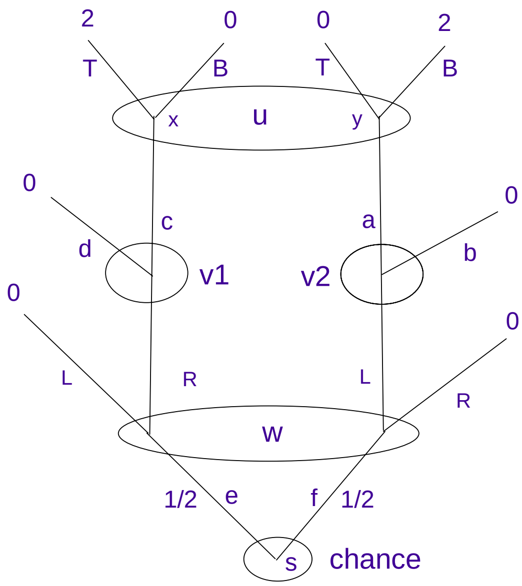

Figure 2

In some games, an information set may not have an earliest-complete-own-predecessor. In the game of Figure 3, for example, u does not have an earliest-complete-own-predecessor. The information set v is an own-predecessor of u since $x \succ v$, but it is not a complete-predecessor since $y \in u$ and y ¨ v. The start node s is an earliest and complete predecessor since all nodes come from s, but it is not the player's "own" information set.

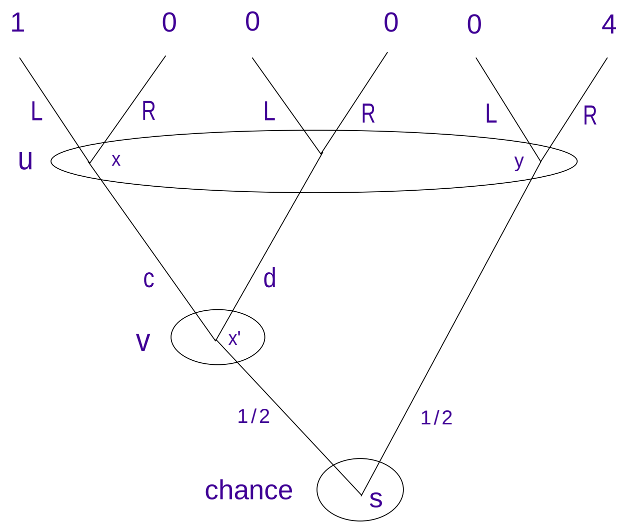

Figure 3

The next Lemma states that for a game with A-loss recall, every information set has an earliest-complete-own-predecessor. The proof is in Section 6.

Lemma 1 If U satisfies A-loss recall, then each $u \in U$ has an earliest-complete-own-predecessor $v \in U_p \cup \{u\}$.

This property will be used to show time-consistent strategies are EA-optimal. Another benefit of Lemma 1 may come in the simplification of large complicated games since it allows a game to be broken up into independent parts. To see this notice that if each information set $u \in U$ has an earliest-complete-own-predecessor, then we can partition U into collections of information sets $U_1, ..., U_k$ such that if two information sets u and v are in the same collection U_i, then they share the same earliest-complete-own-predecessor. This is shown in the game of Figure 4 where U is partitioned into $U1 = \{u1, u2, u3\}$ with the earliest-complete-own-predecessor $u1$ and $U2 = \{v1, v2, v3\}$ with the earliest-complete-own-predecessor $v1$.

When A-loss recall is not satisfied, we may not be able to partition the information sets as described above. For example, in the game of Figure 3, u does not have an earliest-complete-own-predecessor and such a partitioning cannot be carried out.

On the other hand, there are games that don't satisfy A-loss recall but still permit such a partitioning. In Figure 4, if we interchange the choices c

and d at node x (but not at y) in information set $v1$, then the game will no longer satisfy A-loss recall. Nonetheless, every information set in U has an earliest-complete-own-predecessor and the same partitioning works.

Finally we mention that A-loss recall is closely related to Dalkey's (1953) notion of "inflation", whereby an information partition is refined by adding information about the player's previous moves. Kaneko and Kline (1995, Theorem 5.A) show that a player with information partition U has A-loss recall if and only if the player with the complete inflation of U has perfect recall.

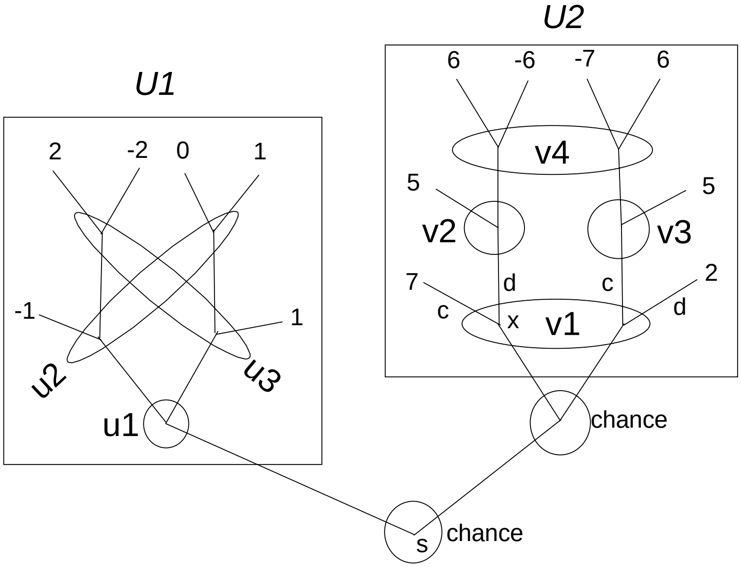

Figure 4

2.2 Time-consistency and EA-optimality

We now describe the EA-optimal and time-consistency views and the associated solution concepts.

The EA-optimality view takes the position that strategies are constructed before the game and then carried out as the game progresses. When the player is given the opportunity to move he consults his strategy and follows what it prescribes. The strategy that will be chosen under this interpretation is likely to be one that maximizes *ex ante* expected utility. Let $p(x, b)$ denote the probability of reaching node x given behavior strategy b is used.

The *ex ante expected utility* of behavior strategy b is given by

$$H(b) = \sum_{z \in Z} p(z, b)h(z).$$

EA-optimality: A strategy $b^* \in B$ is *EA-optimal* iff it maximizes *ex ante* expected utility, i.e., $H(b^*) \geq H(b)$ for all $b \in B$.

We now turn attention to time-consistency. Under this view, the player starts with a strategy, but each time his information set is reached he will decide whether or not to change the strategy. If he changes his strategy, then when he reaches a future information set he will consult the updated strategy and decide whether to make further changes. We assume that the player has access to his updated strategy and a copy of the game tree at each of his information sets. Since the player now has the option to change his strategy at each information set, we might expect something different from an EA-optimal strategy.

Piccione and Rubinstein (1997) suggest a time-consistent strategy will be used in this situation. A time-consistent strategy is one that the player will never choose to change as he moves through the game. The player will choose to change his strategy at an information set u if another strategy yields him a higher conditional expected utility there. The conditional expected utility at an information set depends on the beliefs of the player, which in turn depend on the strategy used previously in the game.

It will be convenient for the purpose of analysis to condition beliefs on subsets of nodes in an information set as well as the entire information set. Let v_j be a subset of nodes of an information set $v \in U$, and let v_i be a subset of the nodes making up v_j. We say that a node x in v_i is *reached* by a strategy b iff $p(x, b) > 0$. The probability that one of the nodes in v_i is reached by strategy b given one of the nodes in v_j is reached by strategy b is formulated as:

$$\mu(v_i \mid v_j, b) = \frac{\sum\limits_{x \in v_i} p(x, b)}{\sum\limits_{x \in v_j} p(x, b)} \tag{2.1}$$

In the case that the subset v_j being conditioned on forms an information set $v \in U$ and the subset v_i consists of a single node x, this leads to the same formulation of beliefs as Piccione and Rubinstein[1] (1997):

$$\mu(\{x\} \mid v, b) = \frac{p(x, b)}{\sum\limits_{x \in v} p(x, b)}. \tag{2.2}$$

The expression $\mu(\{x\} \mid v, b)$ is interpreted as the belief that the player is at node x conditioned on him being at information set v containing x and having used strategy b.

For nodes x and y with $x \succ y$ and any strategy b' we let $p(x \mid y, b')$ denote the probability of reaching node x given we are at y and b' is used in the continuation of the game from x. For a node x, let Z_x denote the set of endnodes following x, i.e., $Z_x = \{z \in Z : z \succ x\}$. Similarly, for an information set u we define $Z_u = \{z \in Z : z \succ u\}$. Then $Z_u = \bigcup\limits_{x \in u} Z_x$. Define $Z_{-u} = Z - Z_u$.

We say that the information set $u \in U$ is *reached* by strategy $b \in B$ iff $p(x, b) > 0$ for some $x \in u$.

[1] Halpern (1997) noticed a potential problem with this formulation when a player is absent-minded. Absent-mindedness is a form of imperfect recall that is discussed in Section 4. We do not want to enter this debate but rather assume beliefs are formulated as stated in equation (2.2).

For an information set u reached by strategy b, the *conditional expected utility* of a strategy b' at u given beliefs $\mu(\{x\} \mid u, b)$ is defined as:

$$H_u(b', \mu(b)) = \sum_{x \in u} \mu(\{x\} \mid u, b) \sum_{z \in Z_x} p(z \mid x, b') h(z). \tag{2.3}$$

Time-consistency: A strategy b is *time-consistent* iff for all information sets $u \in U$ reached by b, and for all $b' \in B$, $H_u(b, \mu(b)) \geq H_u(b', \mu(b))$.

The notion of time-consistency is only applied to information sets reached by the strategy b. In this sense it is weaker than the notions of subgame perfection and perfection (Selten (1975)) which require maximization even at information sets not reached by b.

As we mentioned earlier, we might want to condition beliefs on subsets of nodes of an information set. In particular, we will want to condition on what we call history equivalent nodes in an information set.

The *history* of node $x \in P$ is the set:

$$H(x) = \{(v, c) : x \succ_c v \text{ and } v \in U\} \tag{2.4}$$

$$\text{Nodes } x, y \in P \text{ are } \textit{history equivalent} \text{ iff } H(x) = H(y). \tag{2.5}$$

We can partition the nodes of an information set into subsets of history equivalent nodes. We have the following result which will be used to show EA-optimality implies time-consistency.

Lemma 2: Let $v \in U$ be given and let v_j be a subset of history equivalent nodes in v. For any strategies $b, b' \in B$ and any node $x \in v_j$ reached by both b and b', we have $\mu(\{x\} \mid v_j, b) = \mu(\{x\} \mid v_j, b')$.

This Lemma is an extension of Selten's result (1975, Lemma 4) that in a game of perfect recall, a player's belief that he is at a particular node in an information set is independent of his own strategy. Since we do not require the player to have perfect recall, we only get the independence result amongst history equivalent nodes. The proof follows the same line of reasoning as Selten's, so we have left it out of this paper.

3 A Minimum Condition on Memory for Equivalence

We are now ready to give the main results of this paper. When perfect recall is satisfied, information becomes finer as the game unfolds. Consequently, letting the player update his strategy in a time-consistent manner must be at least as good an option for him. However, since the player has complete information about the game tree it turns out that updates will be irrelevant. The player *ex ante* correctly anticipates what he will do later on down the track and chooses accordingly. This generates an equivalence between time-consistency and EA-optimality for games with perfect recall.

Piccione and Rubinstein showed that the equivalence can be extended to games of imperfect recall where a player never forgets what he learned but might forget what he did (1997, Proposition 2). Players in such games are said to have occurrence memory. This condition was introduced by Okada (1987) and requires more from a player's memory than A-loss recall, but less than perfect recall.

Occurrence Memory: The player has *occurrence memory* iff for any $u, v \in U$ and any $x, y \in u$, $x \succ v$ implies $y \succ v$.

The occurrence memory condition is obtained from the perfect recall condition by eliminating the subscript c of $\succ_c$. Therefore, every player with perfect recall has occurrence memory. As suggested above, every player with occurrence memory also has A-loss recall. To see this, notice that if the player has occurrence memory and $x \succ_c v$, then either $y \succ_c v$ and thus (1) of A-loss recall is satisfied, or $y \succ_d v$ for some choice $d \in A_v$ distinct from c and thus (2) of A-loss recall is satisfied.

The next result is that A-loss recall is the limit on memory for equivalence.

Theorem 1: Let $\Gamma(\cdot) = (K, P, U, C, Z, \rho, \cdot)$ be a game structure defined excluding the payoff assignment. The following two statements are equivalent:

(a) the player has A-loss recall.

(b) for each payoff assignment h, the set of EA-optimal and time-consistent strategies coincide in the game $\Gamma(h)$.

This theorem has two contributions. One is expressed as the sufficiency for equivalence between EA-optimality and time-consistency. By sufficiency we mean the implication of (b) by (a). The second contribution is expressed as necessity for the same equivalence by which we mean the implication of (a) by (b).

The implication of (a) by (b) is equivalent to the statement:

Whenever A-loss recall is violated, there is a payoff assignment h for which the EA-optimal and time-consistent strategies do not coincide.

For each game without A-loss recall, we show how to construct a payoff assignment h under which every EA-optimal strategy is time-inconsistent. Consequently, we actually prove that the set of EA-optimal and time-consistent strategies are disjoint. This part of the theorem still allows the possibility that EA-optimal and time-consistent strategies may coincide for some payoff assignment. For example, if all outcomes yield the same payoff, then every strategy is both time-consistent and EA-optimal regardless of the information structure. Thus we find that for the necessity part, we need to choose payoffs that make the player's forgetfulness payoff relevant.

Consider, as an example of the necessity result, the player in the Game of Figure 3 who does not have A-loss recall. His forgetfulness at information set u is definitely payoff relevant under the payoff assignment of that game. If he could recall that v was reached and he chose c there, then he would know he is at x and clearly want to choose L. Alternatively, if he could recall that v was not reached, then he would know he is at y and want to choose R. The problem is that his forgetfulness makes it impossible for him to distinguish between the two situations when he is at u.

Letting $b_v(c)$ denote the probability of choosing c at v and letting $b_u(L)$ denote the probability of choosing L at u, the *ex ante* expected utility of the player is $\frac{1}{2}b_v(c)b_u(L) + 2(1 - b_u(L)) = \frac{b_u(L)}{2}(b_v(c) - 4) + 2$. Since $b_v(c) \leq 1$, every EA-optimal strategy must use $b_u(L) = 0$, that is, the player chooses R at u for certain. Essentially, he chooses R always because he is at least as likely to be at y as at x, and he gets a much higher payoff from choosing R at y rather than choosing L at x. This strategy cannot be time-consistent since at v he will know that y cannot be reached and thus his best option is to switch to $b_v(c) = b_u(L) = 1$.

4 Existence of time-consistent Strategies

We now turn our attention to the question of existence of a time-consistent strategy. The game of Figure 3 does not have a time-consistent strategy. To see this notice that we just argued that time-consistency at v requires $b_v(c) = b_u(L) = 1$. Next consider time-consistency at u. Chance ensures y is reached with probability $\frac{1}{2}$ regardless of the player's behavior strategy. Thus, the player's belief that he is at y will equal $\frac{1}{2}$. Since his belief he is at x will be at most $\frac{1}{2}$, his conditional expected utility at u will be maximized by setting $b_u(L) = 0$. But this contradicts our finding that time-consistency at v requires $b_u(L) = 1$. Consequently, we conclude that a time-consistent strategy does not exist in this game. Battigalli (1997, Figure 1) noticed this type of problem.

EA-optimal strategies, unlike time-consistent strategies, always exist. Since the behavior strategy space is compact and the *ex ante* payoff function is a continuous function of the player's own strategy, we have the existence of an EA-optimal strategy for any game. This fact is stated as Lemma 3.

Lemma 3. An *ex ante* optimal strategy b^* exists.

Using Lemma 3 and Theorem 1, we obtain the existence of a time-consistent strategy for games with A-loss recall.

To give the main result of this section we need the concept of a conscious-minded player.

Conscious-Minded: A player is *conscious-minded* iff x ¨ y for all $x, y \in u$ and $u \in U$.

This condition restricts our analysis to games where the path of play does not intersect the same information set more than once. Kuhn (1953) introduced this condition and restricted his analysis to games with conscious-minded players. Isbell (1957) and Piccione and Rubinstein (1997) analyzed games with players who are not conscious-minded. Piccione and Rubinstein used the term absent-minded to describe players who are not conscious-minded. The player in the game of Figure 5 is absent-minded.

The main result of this section is that A-loss recall is the limit for existence of time-consistent strategies for all payoff assignments when the player is conscious-minded.

Theorem 2 Let $\Gamma(\cdot) = (K, P, U, C, Z, \rho, \cdot)$ be a game structure defined excluding the payoff assignment and suppose the player is conscious-minded. The following two statements are equivalent:

(a) the player has A-loss recall.

(b) for each payoff assignment h, a time-consistent strategy exists in the game $\Gamma(h)$.

The implication of (b) by (a) in Theorem 2 expresses the existence of a time-consistent strategy for any game of A-loss recall. The implication of (a) by (b) expresses the non-existence of a time-consistent strategy for some payoff assignment whenever a conscious-minded player does not have A-loss recall. Again, as with the implication of (a) by (b) in Theorem 1, we need to make the forgetfulness be payoff relevant.

Let's see what leads to the non-existence result. Under the time-consistency view the player gains some information *while playing the game*. If he can use this information to update his strategy in the continuation of the game he might benefit. For example, in the game of Figure 3 the player will learn at v that he cannot get the high payoff of 4 and thus he should maximize his effort on getting the best payoff left, which is 1. Different realizations of chance relay different pieces of information to the player *while playing the game*. If two different pieces of information lead to two different best plans in the continuation of the game and the continuation paths intersect, then we have a conflict. When such a conflict exists, we will not be able to find a time-consistent strategy. This is precisely what happened in the game of Figure 3.

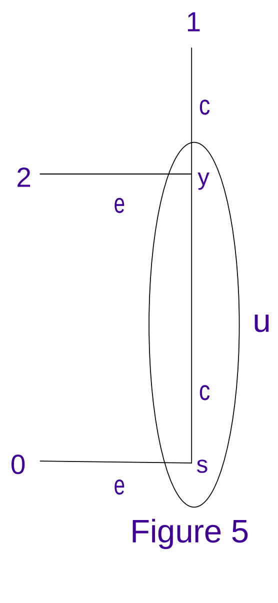

Figure 5

If the player is absent-minded, we may not obtain the implication of (a) by (b) in Theorem 2. The absent-minded driver game of Piccione and Rubinstein (1997) is presented as Figure 5 here. In that game structure, every payoff assignment has a time-consistent strategy even though the player does not satisfy A-loss recall.[2]

We end this section with the following lemma which describes the relationship between all the conditions on memory used in this paper.

Lemma 4. (a) Perfect recall implies Occurrence Memory.

(b) Occurrence Memory implies A-loss recall.

(c) A-loss recall implies Conscious-minded.

Proof of Lemma 4. Parts (a) and (b) were already explained after the definition of occurrence memory. We prove the contrapositive of (c). Suppose that the player is absent-minded. Then we have $u \in U$ and $x, y \in u$ with $x \succ y$. Find the node $y' \in u$ which comes before x and has no predecessors in u. Such a node exists since $y \in u$ and the tree is finite. Then letting $u = v$ we have $x, y' \in u$ and $x \succ_c v$. We also have y' ¨ v since y' has no predecessors in v. Finally, since $x \succ y'$, it follows that for all $w \in U$, if $x \succ_d w$ and $y' \succ_e w$ then $d = e$. ¤

5 Discussion of Theorem 1

In this section we use some results due to Kuhn (1953) and Kline and Kaneko (1995) to give an intuitive explanation of Theorem 1.

First consider why EA-optimality implies time-consistency in a game with A-loss recall. We will use the fact that under perfect recall all EA-optimal strategies are time-consistent. Suppose that the player's EA-optimal strategy is a pure strategy. A behavior strategy b is a *pure strategy* iff for each information set $u \in U$ there is a choice $a_u \in A_u$ such that $b_u(a_u) = 1$. In the game of Figure 2, for example, the EA-optimal strategy b given by $b_w(R) = b_{v1}(c) = b_{v2}(a) = b_u(T) = 1$ is a pure strategy. Since the game satisfies A-loss recall, every loss of memory can be traced back to a loss of memory of the player's own previous action. Thus, if a player with A-loss recall is using a pure strategy, it follows that he will be able to work out everything he forgot by consulting his strategy. For example, under the given EA-optimal strategy for the game of Figure 2, the player will be able to work out at u that the chance move was e. Since his pure strategy suggests a choice of R at w, he concludes that if f were chosen by chance then he would never have reached u. When the player chose a pure EA-optimal strategy before the game he knew that his future self would be able to work out everything he forgot by his strategy and act as-if he has perfect recall. Hence, just as if the player had perfect recall, there is no need for updates of an EA-optimal plan.

[2]On a technical level, the proof of non-existence uses the fact that for a conscious-minded player, every violation of A-loss recall involves two distinct information sets.

What if the player's EA-optimal strategy involves randomization? In this case his strategy may not provide him with enough information to work out everything he forgot even if he has A-loss recall. In the game of Figure 2 if the player chooses to randomize at w, then when he arrives at u he will not be able to work out what he forgot. It turns out, however, that if a player with A-loss recall ever chooses optimally to randomize at an information set, then any forgetfulness that could have been avoided by using a pure strategy must not be relevant to his payoff maximization.

The reasoning is a bit detailed but runs as follows. Kuhn (1953) showed that in a game with A-loss recall every behavior strategy has a payoff equivalent mixed strategy.[3] We present Kuhn's result as Lemma 6.1. A mixed strategy is a probability distribution over the set of pure strategies.[4] It is well known that every mixed strategy has a payoff no greater than some pure strategy assigned a positive probability by that mixed strategy. We present this second result as Lemma 6.2. By these two results we obtain the result in a game with A-loss recall that for any EA-optimal behavior strategy there is a payoff equivalent pure strategy. It follows from the results of Kaneko and Kline (1995) that the payoff equivalent pure strategy would give a player with A-loss recall the payoff maximization potential of the same player with perfect recall (Lemma 6.3 here). So if a player with A-loss recall chooses optimally to follow a non-pure behavior strategy, then it must be true that his forgetfulness is not relevant to his utility maximization.

When we enter the time-consistency framework, the same reasoning will apply. At each information set, if the player cannot work out what he forgot by his strategy, then it must not be relevant to his current and future decisions. If it were relevant, the player would have chosen a pure strategy *ex ante*.

The results used in the arguments just presented are summarized in Lemma 5 below. To present Lemma 5 we need to introduce some concepts which are due to Kuhn (1953). Let $\Gamma[\cdot] = (K, P, \cdot, C, Z, p, h)$ be an extensive game structure defined excluding the player's information partition U. For a given partition U of P, we obtain an extensive game $\Gamma[U] = (K, P, U, C, Z, p, h)$.

The *perfect recall refinement* U_p of U is defined to be the coarsest refinement of U that satisfies perfect recall. This refinement was introduced by Dubey and Kaneko (1982) where they showed that each U has a unique perfect recall refinement. The perfect recall refinement U_p of U can be used to define a new extensive game $\Gamma[U_p]$. Indeed, if U itself satisfies perfect recall, then $U = U_p$ and thus $\Gamma[U] = \Gamma[U_p]$. Since U_p is a refinement of U the strategies defined on U_p can also be defined on U. But the converse will not be the case if U does not satisfy perfect recall. Given the game structure $\Gamma[\cdot]$, let U and V be two partitions of P. We say that strategy b for $\Gamma[U]$ is *realization equivalent* to

[3] Actually, he showed the result for all games with conscious-minded players. Conscious-mindedness is defined in Section 4 of this paper where we show that all players with A-loss recall are conscious-minded.

[4] Notice that mixed and behavior strategies are quite different objects since the former involves randomization over pure strategies, while the later involves randomization at each information set. For more on these differences and their implications see Kuhn (1953), Selten (1975), and Kaneko and Kline (1995).

strategy b' for $\Gamma[V]$ iff $p(x, b) = p(x, b')$ for all $x \in P \cup C \cup Z$. The notion of realization equivalence allows us to compare a strategy from U with a strategy from U_p. If a strategy b for U is realization equivalent to a strategy b_p for U_p then clearly $H(b) = H(b_p)$, i.e., the two strategies are payoff equivalent.

We have the following Lemma.

Lemma 5. Let the player have A-loss recall in the game $\Gamma[U]$ and let $\Gamma[U_p]$ be the game defined by the perfect recall refinement U_p of U. The following two statements hold:
(a) If a behavior strategy b^* for $\Gamma[U]$ is EA-optimal in $\Gamma[U]$, then every realization equivalent behavior strategy b^*_p for $\Gamma[U_p]$ is EA-optimal in $\Gamma[U_p]$.
(b) There is a pure strategy b^* for $\Gamma[U]$ such that b^* is EA-optimal in $\Gamma[U]$.

Now let's see why time-consistency implies EA-optimality under the condition of A-loss recall. It suffices to consider only the player's "earliest information sets" (those closest to the root node) since time-consistency requires, among other things, that the player does not want to update at any of these information sets. These earliest information sets are precisely the earliest-complete-own-predecessors defined in Section 2. Lemma 1 guarantees that the beliefs at any earliest-complete-own-predecessor u_1 will be independent the player's own strategy, and that if some information set u of the player follows u_1 then u does not follow any other earliest-complete-own-predecessor. This allows us to separate the continuation paths of the player's information sets at his "earliest information sets". By such a separation it is easy to see that maximization of conditional expected utility at each "earliest information set" implies EA-optimality. For example, in the game of Figure 4, time-consistency at $u1$ and $v1$ guarantees EA-optimality.

Finally, let's see why the equivalence fails when A-loss recall is violated. Whenever A-loss recall is violated the player will sometimes not be able to work out from a pure strategy everything that he forgot. For example, in the game of Figure 3, when the player arrives at u he will not be able to work out by any pure strategy whether or not he reached v. Here information truly becomes coarser as we move through the tree. Consequently, the player is able to improve on his conditional expected utility by updating his strategy in the game sometime after information became finer, but before the coarsening occurs. In the game of Figure 3 the player gains from updating at v. Thus we see that the divergence between time-consistency and EA-optimality comes only when information is coarsened as we move through the game and the use of a pure strategy would not allow the player to deduce what he forgot.

Earlier in the paper we mentioned that perfect recall implies information becomes finer as we move through the game tree. In this case there is no discrepancy between time-consistency and EA-optimality. When we move beyond perfect recall, information may become coarser as we move through the tree. It is the fact that information becomes coarser that brings about a potential discrepancy between time-consistency and EA-optimality. Theorem 1 shows that these discrepancies do not arise until we move beyond A-loss recall even though information clearly becomes coarser in the region between perfect recall and A-loss recall. What A-loss recall does is allow the player to work out by his strategy what he forgot whenever it is relevant. So effectively, information does not become coarser as we move through the tree.

6 Proofs of Theorems

The reader will notice that the notion of a mixed strategy is used in some of the proofs of this section. The current paper is about behavior strategies and the use of mixed strategies is purely for technical convenience. We could have avoided mixed strategies altogether but this would have lengthened the proofs. Instead, we exploit some similarities between mixed and behavior strategies that have been proved by other authors starting with H. Kuhn (1953).

Let $\Pi \subset B$ denote the set of pure strategies. Recall that a pure strategy assigns probability 1 to some choice $c \in A_u$ for each information set $u \in U$. A *mixed strategy* is a probability distribution over Π.

The following Lemmas are used in the proofs of Theorem 1 and Lemma 5.

Lemma 6.1 (Kuhn (1953)): If the player is conscious-minded, then every behavior strategy has a realization equivalent[5] mixed strategy.

Lemma 6.2 For a mixed strategy q let $\Pi(q)$ be the set of pure strategies chosen with positive probability by q. Then $\max_{\pi \in \Pi(q)} H(\pi) \geq H(q)$.

Lemma 6.3 Let a player with the information partition U have A-loss recall and let U_p be the perfect recall refinement of U. Every mixed strategy for $\Gamma[U_p]$ has a realization equivalent mixed strategy for $\Gamma[U]$ and every mixed strategy for $\Gamma[U]$ has a realization equivalent mixed strategy for $\Gamma[U_p]$.

One proof of Lemma 6.1 is presented in Kaneko and Kline (1995, Theorem 2A). Lemma 6.2 is a result of the payoff from a mixed strategy being a weighted average of the payoffs from pure strategies. Lemma 6.3 is proved in Kaneko and Kline (1995, Theorem 5A).

The reader will notice that we prove Theorem 2 before the converse of Theorem 1. The reason is that Theorem 2 is used to prove that converse.

6.1 Proof of Lemma 1

Let the player have A-loss recall and consider $u \in U$. First we show that there is a $v \in U_p \cup \{u\}$ satisfying $V_p = \emptyset$.

Suppose the contrary, i.e.,

$$\text{for every } v \in U_p \cup \{u\}, \text{ there is a } w \in V_p. \tag{6.1}$$

Then choose:

$$x \in u,\ v \in U_p, \text{ and } x' \in v \text{ such that } x \succ x', \text{ and } [w \in V_p \ \Rightarrow\ x' \ \ddot{}\ w]. \tag{6.2}$$

[5]Conscious-minded is defined in Section 4 and realization equivalence is defined in Section 5 a few paragraphs before Lemma 5.

Such x, x' and v can be found since: (6.1) holds, and the game is finite. First, (6.1) guarantees a $v \in U_p$. Finiteness allows us to choose the node x' of the player preceding x that is closest to the root of the tree and use the information set v containing x'. We fix x, x' and v from (6.2) in what follows.

Since $v \in U_p$ we have by (6.1) that there is a $w \in V_p$. We fix such a w in what follows. Since x' ¨ w (by (6.2)), there must be a $y \in v$ distinct from x' satisfying $y \succ w$. So we have $x', y \in v$ and $y \succ w$ and x' ¨ w. Hence, by A-loss recall there is a $w' \in U$ with $x' \succ_d w'$ and $y \succ_e w'$ for $d \neq e$. But this contradicts (6.2) since $w' \in V_p$.

Next we show that such a $v \in U_p$ satisfying $V_p = \emptyset$ and $v \neq u$ must satisfy $x \succ v$ for all $x \in u$ which will complete the proof. To do this we use the following claim.

Claim 1 Let the player have A-loss recall. If there are $u, v \in U$ and $x, y \in u$ such that $x \succ v$ and y ¨ v, then there is a $w \in V_p$ such that $x \succ_d w$ and $y \succ_e w$ for distinct $d, e \in A_w$.

Proof of Claim 1: Suppose $u, v \in U$ and $x, y \in u$ such that $x \succ v$ and y ¨ v.

Then by A-loss recall there is a $w \in U$ with $x \succ_d w$ and $y \succ_e w$ for distinct $d, e \in A_w$. Hence, it suffices to show that some such w is in V_p. Find the "earliest" such w, which we know exists since the game is finite. We show that this w is contained in V_p.

Suppose to the contrary that the "earliest" such w is not contained in V_p. Then since $x \succ v$, $x \succ w$, y ¨ v, $y \succ w$, and v ¨ w, we have $x', y' \in w$ satisfying $x \succ x'$, $y \succ y'$, and most importantly $x' \succ v$ and y' ¨ v. By the choice of w as the "earliest", there is no $w' \in U$ with $x' \succ_f w'$ and $y' \succ_g w'$ for distinct $f, g \in A_w$. But this contradicts A-loss recall. ¤

Now to complete the proof using Claim 1 suppose that $v \in U_p$ satisfies $V_p = \emptyset$ and $v \neq u$. Then there is an $x \in u$ and a $c \in A_v$ such that $x \succ_c v$. Consider $y \in u$. By A-loss recall either (1) $y \succ_c v$ or (2) there is a $w \in U$ such that $x \succ_d w$ and $y \succ_e w$ for $d \neq e$. If (1) holds then obviously $y \succ v$. If (2) holds, then by Claim 1 and the fact that $V_p = \emptyset$, the w in (2) must satisfy $w = v$ and thus again $y \succ v$. ¤

6.2 Proof of Theorem 1: (a) ⇒ (b):

We fix the payoff assignment h and prove that in the game $\Gamma(h)$: (i) b is time-consistent implies b is EA-optimal, and (ii) b is EA-optimal implies b is time-consistent.

Proof of (i): Let $U_e \subseteq U$ denote the set of "earliest information sets" of the player, i.e., $u \in U_e$ iff $U_p = \emptyset$. Recall that $U_p = \{v \in U : u \succ v\}$. For each information set $u \in U_e$ let $Z_u = \{z \in Z : z \succ u\}$. Define $Z_U \equiv \bigcup_{u \in U_e} Z_u$ and $Z_{-U} \equiv Z - Z_U$. Then making use of Lemma 1, we can write the expected payoff as:

$$H(b) = \sum_{u \in U_e} \sum_{x \in u} p(x, b) H_u(b, \mu(b)) + \sum_{z \in Z_{-U}} p(x, b) h(z). \tag{6.3}$$

Notice that for each $u \in U_e$, if $x \in u$ then $p(x,b)$ is independent of b. Since time-consistency requires that b maximizes $H_u(b,\mu(b))$ at each $u \in U_e$, we have by (6.3) that b maximizes $H(b)$ and is thus EA-optimal.

Proof of (ii): We prove the contrapositive. Suppose b is time-inconsistent. We fix such a time-inconsistent b in what follows and show that b is not EA-optimal using Claims 1 to 6.

Claim 1: (a) There is a mixed strategy q_b that is realization equivalent to b; and
(b) either: (1) b is not EA-optimal or (2) $H(b) = H(\pi)$ for every pure strategy π assigned positive probability by q_b.

Proof of Claim 1: (a) The existence of a realization equivalent mixed strategy q_b is guaranteed by Lemma 6.1 since A-loss recall implies conscious-minded (Lemma 4 (c)). (b) follows from Lemma 6.2. ¤

Suppose (1) of Claim 1 (b) holds. Then the entire proof of (ii) is complete.

We assume throughout the following that (2) of Claim 1 (b) holds. Then since b is time-inconsistent:

$$\text{there is a } v \in U \text{ and some } b' \neq b \text{ such that } H_v(b',\mu(b)) > H_v(b,\mu(b)). \quad (6.4)$$

We fix v and b' and b from (6.4) in what follows.

We will consider one subset of nodes $v_i \subseteq \{x \in P : x \in v\}$ to be chosen below. Based on that subset and the strategies b and b' defined above, we construct two new strategies a^* and b^*. We show that $H(b^*) > H(a^*) = H(b)$ which implies that b cannot be EA-optimal.

Let $V_{he} = \{v_1, ..., v_k\}$ denote the partition of v into subsets $v_1, ..., v_k$ of history equivalent nodes.[6]

We can express the conditional expected utility $H_v(a',\mu(a))$ when beliefs at v are formed by some a satisfying $\sum_{x\in v} p(x,a) > 0$ and a' is played in the continuation of the game as:

$$H_v(a',\mu(a)) = \sum_{v_j \in V_{he}} \mu(v_j \mid v, a) H_{v_j}(a',\mu(a)) \quad (6.5)$$

where: $H_{v_j}(a',\mu(a)) =$

$$\begin{cases} \sum_{x\in v_j} \mu(\{x\} \mid v_j, a) \sum_{z\in Z_x} p(z \mid x, a')h(z) \text{ if } \sum_{x\in v_j} p(x,a) > 0 \\ 0 \text{ if } \sum_{x\in v_j} p(x,a) = 0. \end{cases} \quad (6.6)$$

Claim 2: There is some $v_i \in V_{he}$ with $\sum_{x\in v_i} p(x,b) > 0$ and $H_{v_i}(b',\mu(b)) > H_v(b,\mu(b))$.

Proof of Claim 2: This follows from (6.4), (6.5) and (6.6). ¤

[6] History Equivalence is defined in (2.7) of Section 2.2.

We fix such a v_i from Claim 2 in what follows. Just as we did with information sets, we write $v_i \succ_c u$ iff there is some node $x \in v_i$ with $x \succ_c u$. We define $V_{ip} = \{u \in U : v_i \succ u\}$ and $V_{if} = \{u \in U : u \succ v_i\}$.

We now construct the strategies b^* and a^*.

Define b^* as follows. For each information set $u \in U$ and each choice $c \in A_u$,

$$b^*_u(c) = \begin{cases} 1 \text{ if } u \in V_{ip} \text{ and } v_i \succ_c u \\ 0 \text{ if } u \in V_{ip} \text{ and } v_i \not\succ_c u \\ b'_u(c) \text{ if } u \in \{v\} \cup [V_{if} - (V_{if} \cap V_{ip})] \\ b_u(c) \text{ if } u \in U - (\{v\} \cup V_{if} \cup V_{ip}). \end{cases} \tag{6.7}$$

Define a^* by:

$$a^*_u(c) = \begin{cases} 1 \text{ if } u \in V_{ip} \text{ and } v_i \succ_c u \\ 0 \text{ if } u \in V_{ip} \text{ and } v_i \not\succ_c u \\ b_u(c) \text{ if } u \in U - V_{ip} \end{cases} \tag{6.8}$$

Both a^* and b^* make v_i be reached with the maximum probability available to the player. The strategy a^* follows b everywhere else. The strategy b^* follows b' at v and at those information sets following v, while following b everywhere else off the path through v.

Claim 3: $H(a^*) = H(b)$.

Proof of Claim 3: By Lemma 4 (c) and Lemma 6.1 there is a mixed strategy q_{a^*} which is realization equivalent to a^* and a mixed strategy q_b which is realization equivalent to b. Hence, $H(a^*) = H(q_{a^*})$ and $H(b) = H(q_b)$. Recall that we are analyzing the case (from Claim 1) where $H(\pi) = H(q_b)$ for each pure strategy π assigned positive probability by q_b. Since a^* only puts positive probability on choices that are chosen with positive probability by b it follows that the set of pure strategies assigned positive probability by q_{a^*} are a subset of those assigned positive probability by q_b. Thus, $H(q_b) = H(q_{a^*})$. ¤

Claims 4 and 5 are used to show that $H(b^*) > H(a^*)$ (Claim 6). This together with Claim 3 proves that b is not EA-optimal.

Claim 4: (a) $H_v(b^*, \mu(b^*)) = H_{v_i}(b^*, \mu(b^*))$,
(b) $H_v(a^*, \mu(a^*)) = H_{v_i}(a^*, \mu(a^*))$,
(c) $H_{v_i}(b^*, \mu(b^*)) = H_{v_i}(b', \mu(b))$.
(d) $H_{v_i}(a^*, \mu(a^*)) = H_{v_i}(b, \mu(b))$.

Proof of Claim 4: Parts (a) and (b) follow from (6.5) and (6.6) since both a^* and b^* make $\mu(v_i \mid v, a^*) = \mu(v_i \mid v, b^*) = 1$.

For part (c) notice that b^* assigns probability 1 to the choices on the path to v_i at every information set in V_{ip}. Hence, since the player satisfies A-loss recall, if $w' \in V_{if}$ and $p(w', b^*) > 0$, then $w' \notin V_{ip}$. This implies that for any $z \in Z_v$ and any $x \in v_i$, $p(z \mid x, b^*)$ is independent of choices at information sets $w \in V_{ip}$. Since b' and b^* coincide on $\{v\} \cup [V_{if} - (V_{if} \cap V_{ip})]$, it follows that $p(z \mid x, b^*) = p(z \mid x, b')$ for each $x \in v_i$ and $z \in Z_{v_i}$. Notice also that by Lemma 2, for a given v_j, $\mu(x \mid v_j, a)$ is independent of a, which means in particular that $\mu(x \mid v_i, b^*) = \mu(x \mid v_i, b')$ for all $x \in v_i$. We thus have (c) using the definition of $H_{v_i}(a, \mu(a))$ given in (6.6). Part (d) follows by a similar argument.¤

Using the partition of endnodes into $Z_v = \{z \in Z : z \succ v\}$ and $Z_{-v} \equiv Z - Z_v$ we can write the expected payoff under any strategy a as:

$$H(a) = \sum_{x \in v} p(x,a) H_v(a, \mu(a)) + [1 - \sum_{x \in v} p(x,a)] H_{-v}(a, \mu(a)) \quad (6.9)$$

$$\text{where} \quad : \quad H_{-v}(a, \mu(a)) = \begin{cases} \frac{1}{[1 - \sum_{x \in v} p(x,a)]} \sum_{z \in Z_{-v}} p(z,a) h(z) \text{ if} \sum_{x \in v} p(x,a) < 1 \\ 0 \text{ if } \sum_{x \in v} p(x,a) = 1. \end{cases}$$

Claim 5: (a) $\sum_{x \in v} p(x,b^*) H_v(b^*, \mu(b^*)) > \sum_{x \in v} p(x,a^*) H_v(a^* \mu(a^*))$,
(b) $[1 - \sum_{x \in v} p(x,a^*)] = [1 - \sum_{x \in v} p(x,b^*)]$,
(c) $\sum_{z \in Z_{-v}} p(z,a^*) h(z) = \sum_{z \in Z_{-v}} p(z,b^*) h(z)$.

Proof of Claim 5: Part (a): Since a^* and b^* both assign positive probability to choices on the path to v if and only if they are on the path to v_i it follows that:

$$\sum_{x \in v} p(x,b^*) = \sum_{x \in v_i} p(x,b^*) = \sum_{x \in v_i} p(x,a^*) = \sum_{x \in v} p(x,a^*). \quad (6.10)$$

Using this result and Claims 2 and 4, we have:

$$\begin{aligned} \sum_{x \in v} p(x,b^*) H_v(b^*, \mu(b^*)) &= \sum_{x \in v_i} p(x,b^*) H_{v_i}(b^*, \mu(b^*)) = \quad (6.11) \\ \sum_{x \in v_i} p(x,b^*) H_{v_i}(b', \mu(b)) &> \sum_{x \in v_i} p(x,b^*) H_{v_i}(b, \mu(b)) = \\ \sum_{x \in v_i} p(x,a^*) H_{v_i}(a^*, \mu(a^*)) &= \sum_{x \in v} p(x,a^*) H_v(a^* \mu(a^*)). \end{aligned}$$

The first equality in (6.11) follows from Claim 4 (a) and (6.10). The second equality follows from Claim 4 (c). The inequality follows from the choice of v_i (see Claim 2). The next equality follows from Claim 4 (d) and (6.10). The last equality follows from Claim 4 (b) and (6.10).

Part (b): This follows directly by (6.10).

Part (c): Since a^* and b^* only disagree on $[\{v\} \cup (V_{if} - (V_{if} \cap V_{ip}))]$ it follows that for any $z \in Z_{-v}$ and any $w \in U$ with $z \succ w$ we have that $a^*_w = b^*_w$. Hence, for each $z \in Z_{-v}$ we have $p(z,a^*) = p(z,b^*)$. From this we obtain the desired result. ¤

Claim 6: $H(b^*) > H(a^*)$.

Proof of Claim 6: Just put (a), (b) and (c) of Claim 5 into the formula (6.9).¤

6.3 Proof of Theorem 2

$(a) \Rightarrow (b)$ follows immediately from Lemma 3 and Theorem 1.

We prove the contrapositive of $(b) \Rightarrow (a)$, i.e., if a conscious-minded player does not have A-loss recall, then a time-consistent strategy does not exist for some payoff assignment h.

The negation of *A-loss recall* can be written as:

$$\begin{aligned} \text{there are } u, v &\in U,\ x, y \in u \text{ and } x' \in v \text{ such that } x \succ_c x',\ y \ddot{\ } v, \text{ and} \\ \text{for all } w &\in U,\ x \succ_d w \text{ and } y \succ_e w \text{ implies } d = e. \end{aligned} \tag{6.12}$$

Since the player is conscious-minded, u and v in (6.12) must be distinct information sets. Figure 3 is an example which may be used to follow the method of proof. Let L and R be distinct choices at u. We fix L, R and x, y, x', u, v, c from (6.12) in the following.

Let $A(y) = \{c \in \bigcup_{\overline{x} \in C} A_{\overline{x}} : y \succ_c \overline{x} \text{ for some } \overline{x} \in C\}$ denote the choices taken by chance leading to node y. Let $P_y = \prod_{\overline{c} \in A(y)} p_{\overline{x}}(\overline{c})$ denote the product of the chance probabilities assigned to choices of nature leading to node y. In the same way we define $P_{x'}$ for x' defined in (6.12). For example, in the game of Figure 3, $P_y = 1/2$ and $P_{x'} = 1/2$.

Consider the payoff assignment h defined by:

$$h(z) = \begin{cases} 1 \text{ if } z \succ_L x \\ 2/P_y \text{ if } z \succ_R y \\ 0 \text{ otherwise.} \end{cases} \tag{6.13}$$

We will use Claim 1 below to show that any time-consistent strategy b^* for this payoff assignment h requires both $b_u^*(L) = 0$ (Claim 2), and $b_u^*(L) = 1$ (Claim 3). Since no behavior strategy can satisfy both claims, a time consistent strategy does not exist.

Claim 1 Every time-consistent strategy b^* satisfies $p(y, b^*) = P_y > 0$ and $p(x', b^*) = P_{x'} > 0$.

Proof of Claim 1: We show that $p(y, b^*) = P_y > 0$. The same method can be used to show $p(x', b^*) = P_{x'} > 0$.

Since (6.12) is satisfied, there is at least one chance node on the path to y and since the chance probability distribution is completely mixed, it follows that $P_y > 0$.

Clearly $p(y, b^*) = P_y$ in the case that there are no information sets of the player on the path to y. Suppose that information sets $w_1, ..., w_k$ of the player are on the path to y ordered from earliest to latest. Since the player is conscious-minded, $w_1, ..., w_k$ and u are distinct. Let $y_1, ..., y_k$ be the respective nodes at those information sets on the path to y, and let $d_1, ..., d_k$ be the respective choices at those information sets that lead to y. Hence we have $y \succ_{d_k} y_k \succ_{d_{k-1}} y_{k-1} \succ_{d_{k-2}} y_{k-2} \cdots y_2 \succ_{d_1} y_1$.

We suppose the inductive hypothesis that for some integer $i \in \{1, ..., k\}$ we have $b_{w_j}(d_j) = 1$ for $j = 1, ..., i-1$. We then show that time-consistency requires $b_{w_i}(d_i) = 1$. By induction we will have $p(y, b^*) = P_y$.

For each w_i, $i \in \{1, ..., k\}$ we have one of two cases:

Case 1: there is an $\widetilde{x} \in w_i$ satisfying $x \succ_e \widetilde{x}$;

Case 2: there is no $\widetilde{x} \in w_i$ satisfying $x \succ_e \widetilde{x}$.

Let b be any strategy satisfying the inductive hypothesis. For any strategy b' used in the continuation of the game from w_i:

$H_{w_i}(b', \mu(b)) =$

$$\begin{cases} \mu(\{y_i\} \mid w_i, b)p(y \mid y_i, b') \left(\frac{2}{P_y}\right) + \mu(\{\widetilde{x}\} \mid w_i, b)p(x \mid \widetilde{x}, b') \text{ in case 1.} \\ \mu(\{y_i\} \mid w_i, b)p(y \mid y_i, b') \left(\frac{2}{P_y}\right) \text{ in case 2.} \end{cases} \quad (6.14)$$

In either case, time-consistency requires that $b'_{w_i}(d_i) = 1$ since this is the only choice that guarantees a positive expected payoff and under the inductive hypothesis $\mu(\{y_i\} \mid w_i, b) > 0$. In case 1, (6.12) implies that the choice at w_i on the path to x is the same choice as d_i on the path to y. ¤

Claim 2 Time-consistency of b^* at u implies $b^*_u(L) = 0$.

Proof of Claim 2: Since there is a positive payoff only if L is chosen at x or R is chosen at y it follows that for any time-consistent strategy b^* and any strategy b' used in the continuation of the game from u we have:

$$H_u(b', \mu(b^*)) = \mu(\{x\} \mid u, b^*)b'_u(L) + \mu(\{y\} \mid u, b^*)b'_u(R)(2/P_y). \quad (6.15)$$

Since time-consistency of b^* requires $p(y, b^*) = P_y$ by Claim 1, it follows that:

$$H_u(b', \mu(b^*)) = \frac{1}{\sum_{\overline{x} \in u} p(\overline{x}, b^*)}[p(x, b^*)b'_u(L) + 2b'_u(R)]. \quad (6.16)$$

Maximization of this expression requires $b'_u(R) = 1$ and thus $b'_u(L) = 0$. ¤

Claim 3: Time-consistency of b^* at v implies $b^*_u(L) = 1$.

Proof of Claim 3: Conditional expected utility at v given a time-consistent strategy b^* to form beliefs at v and a strategy b' used in the continuation of the game is:

$$H_v(b', \mu(b^*)) = \mu(\{x'\} \mid v, b^*)p(x \mid x', b')b'_u(L). \quad (6.17)$$

Since $p(x', b^*) > 0$ by Claim 1, we have $\mu(\{x'\} \mid v, b^*) > 0$. Also, since $p(x \mid x', b')$ is independent of b'_u and can be made positive by a completely mixed strategy, it follows that (6.17) is maximized only if $b'_u(L) = 1$. ¤ ¤

6.4 Proof of Theorem 1: (b) $\Rightarrow$ (a)

We show that if A-loss recall is not satisfied, then there is a payoff assignment for which every EA-optimal strategy is time-inconsistent.

Recall that the negation of *A-loss recall* can be written as:

$$\begin{aligned} \text{there are } u, v &\in U,\ x, y \in u \text{ and } x' \in v \text{ such that } x \succ_c x',\ y \ddot{\ } v, \text{ and} \\ \text{for all } w &\in U,\ x \succ_d w \text{ and } y \succ_e w \text{ implies } d = e. \end{aligned} \quad (6.18)$$

Case 1: The player is conscious-minded and thus the u, v in (6.18) are distinct. Figure 3 is an example.

It follows by Theorem 2 ((b) $\Rightarrow$ (a)) that there is a payoff assignment h for which a time-consistent strategy does not exist. By Lemma 3, an EA-optimal strategy exists for h and the proof is complete.

Case 2: The player is absent-minded and thus there are u and v satisfying (6.18) where $u = v$. Figure 5 is an example.

Since u and v in (6.18) coincide and the game is finite:

$$\text{there are } x, x' \in u \text{ such that } x \succ_c x', \text{ and} \tag{6.19}$$

$$[x'' \in u \text{ and } x'' \neq x'] \text{ implies } [x' \;\ddot{}\; x'' \text{ and } x \;\ddot{}\; x'']. \tag{6.20}$$

Condition (6.20) means we have chosen x and x' in (6.19) so that x' has no predecessors in u and x' is the only predecessor of x in u. This can always be done since the game is finite. Let d be some choice in A_u different from c. We fix $x, x', y, u = v, c$ and d in what follows. Consider the following payoff assignment:

$$h(z) = \begin{cases} 1 \text{ if } z \succ_c x \\ 2 \text{ if } z \succ_d x \\ 0 \text{ otherwise.} \end{cases} \tag{6.21}$$

We will show that for the given payoff assignment h, every EA-optimal strategy is time-inconsistent.

For a strategy b, define:

$$p(x \mid x', b_{-u}) \equiv \frac{p(x \mid x', b)}{b_u(c)}. \tag{6.22}$$

By the choice of x and x' in (6.19) to satisfy (6.20), it follows that $p(x \mid x', b_{-u})$ is independent of b_u.

In Claim 1 we show that every EA-optimal strategy b^* for the given payoff assignment satisfies $b^*_u(c) = 1$. Then we show that such a strategy is time-inconsistent at u.

Claim 1: If b^* is EA-optimal for $\Gamma(h)$ then $p(x, b^*) > 0$ and $b^*_u(c) = 1$.

Proof of Claim 1: Since $p(x, b) = p(x \mid x', b)p(x', b)$ by definition, it follows by (6.22) that:

$$p(x, b) = p(x', b)p(x \mid x', b_{-u})b_u(c) \tag{6.23}$$

Using (6.23), the payoff for any strategy b given h defined in (6.21) can be written as:

$$H(b) = p(x', b)p(x \mid x', b_{-u})[(b_u(c))^2 + 2b_u(c)b_u(d)] \tag{6.24}$$

Since the player can obtain $H(b) > 0$ by a completely mixed strategy and $H(b) = 0$ if $p(x, b) = 0$ it follows that $p(x, b^*) > 0$. Next, notice that $p(x', b)p(x \mid x', b_{-u})$ is independent of b_u by the choice of x and x' in (6.19) to satisfy (6.20). Hence, every EA-optimal strategy b^* must have $b^*_u(c) = 1$. ¤

Next, consider an EA-optimal strategy b^* and any strategy b used in the continuation of the game from u. Let $x, y_1, ..., y_k$ be the $k+1$ nodes in u that come from x' by taking choice c every time u is reached. We have :

$$\begin{aligned} H_u(b, \mu(b^*)) &= \mu(\{x'\} \mid u, b^*)p(x \mid x', b_{-u})[(b_u(c))^2 + 2b_u(c)b_u(d)] + \\ \mu(\{x\} &\mid u, b^*)[b_u(c) + 2b_u(d)] + \sum_{i=1}^{k} \mu(\{y_i\} \mid u, b^*). \end{aligned} \tag{6.25}$$

By Claim 1 that $p(x, b^*) > 0$ and the fact that $x \succ x'$, it follows that $\mu(\{x'\} \mid u, b^*) \geq \mu(\{x\} \mid u, b^*) = \frac{p(x,b^*)}{\sum_{\overline{x} \in u} p(\overline{x},b^*)} > 0$. By these facts and the fact that $p(x \mid x', b_{-u})$ lies in $[0, 1]$ and does not depend on b_u, it follows that (6.25) is maximized by choosing $b_u(c) < 1$. Hence, by Claim 1, every EA-optimal strategy b^* is time-inconsistent. ¤

6.5 Proof of Lemma 5

We prove the contrapositive of (a). Consider a game $\Gamma[U]$ with A-loss recall. Let $\Gamma[U_p]$ be the game based on the perfect recall refinement of U. Suppose that b_p^* for $\Gamma[U_p]$ is realization equivalent to b^* for $\Gamma[U]$ and b_p^* is not EA-optimal in $\Gamma[U_p]$. Then $H(b^*) = H(b_p^*) < H(b_p')$ for some other strategy b_p' for $\Gamma[U_p]$. Consider a mixed strategy q for $\Gamma[U_p]$ that is realization equivalent to b_p'. Such a strategy exists by Lemma 6.1. By Lemma 6.3 there is a mixed strategy q' for $\Gamma[U]$ that is realization equivalent to the mixed strategy q for $\Gamma[U_p]$. Hence, $H(q') = H(q) = H(b_p') > H(b^*)$. Finally, by Lemma 6.2 we have a pure strategy π for $\Gamma[U]$ that is chosen with positive probability by q' and satisfies $H(\pi) \geq H(q')$, and thus by the previous string of equalities and an inequality we have that $H(\pi) > H(b^*)$. Since π is also a behavior strategy for $\Gamma[U]$ it follows that b^* cannot be optimal in $\Gamma[U]$.

Part (b) follows by Lemma 4 (c), Lemma 6.1 and Lemma 6.2. ¤

7 Conclusions

In this paper we explored the relationship between time-consistent strategies and EA-optimal strategies. What is at issue here is the player's ability to update his strategy. We showed that A-loss recall (due to Kaneko and Kline (1995)) guarantees equivalence between EA-optimal and time-consistent strategies. A-loss recall is weaker than perfect recall since it allows a player to forget things he learned and did. However, it is stronger than absent-mindedness since it requires that every memory loss can be traced back to some memory loss of the player's own action.

When a player does not have A-loss recall, the equivalence between EA-optimal and time-consistent strategies breaks down for some payoff assignment. In these cases, the ability for a player to update his strategy becomes relevant. The potential non-equivalence between EA-opitmal and time-consistent strategies has been referred to as the absent-minded driver's paradox. It was viewed as a paradox because, in spite of apparently no new information, a player wants to update his strategy. Since A-loss recall is stronger than absent-mindedness, the results of the present paper suggest that the absent-minded driver's paradox is not caused by absent-mindedness but rather by a lack of A-loss recall.

Finally, by concentrating our efforts on conscious-minded players (those who are not absent-minded), we were able to show that A-loss recall is also a necessary and sufficient condition for the existence of a time-consistent strategy for all payoff assignments.

References

[1] Aumann, R., S. Hart, and M. Perry (1997), The absent-minded driver, *Games and Economic Behavior,* **20**, 102-116.

[2] Battigalli, P., Dynamic consistency and imperfect recall (1997), *Games and Economic Behavior,* **20**, 31-50.

[3] Dalkey N. (1957), Equivalence of information patterns and essentially determinate games, in Contributions to the Theory of Games II, Kuhn, H. W., Tucker A. W. (eds), 217-243, Princeton University Press.

[4] Dubey P., and M. Kaneko (1982), Information about moves in extensive games: I, Cowles Foundation Discussion Paper 625.

[5] Grove, A. and J. Halpern (1997), On the expected value of games with absentmindedness, *Games and Economic Behavior,* **20**, 51-65.

[6] Halpern, J. (1997), On ambiguities in the interpretation of game trees, *Games and Economic Behavior*, **20**, 66-96.

[7] Kaneko, M., and J. Kline (1995), Behavior Strategies, Mixed Strategies and Perfect Recall, *International Journal of Game Theory*, **24**, 127-145.

[8] Kuhn, H.W. (1953), Extensive games and the problem of information, in Contributions to the Theory of Games II, Kuhn, H. W., and A. W. Tucker (eds), 193-216, Princeton University Press.

[9] Isbell, J. (1957), Finitary Games, in Contributions to the Theory of Games III, 79-96, Princeton University Press.

[10] Okada, A. (1987), Complete inflation and perfect recall in extensive games. *International Journal of Game Theory*, **16**, 85-91.

[11] Piccione, M., and A. Rubinstein (1997), On the interpretation of decision problems with imperfect recall, *Games and Economic Behavior*, **20**, 3-24.

[12] Selten, R. (1975), Reexamination of the perfectness concept for equilibrium points in extensive games, *International Journal of Game Theory*, **4**, 25-55.