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Spreadsheet Numerical Modeling in Secondary School Physics and Biology

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Abstract

The paper gives three examples of numeric modeling with spreadsheets in secondary school (age 15+) physics and biology – free fall in the air, animal population, and damped oscillation. The aim is to introduce the reader to numeric modeling. The simplest numeric methods are used – Euler's, and the finite difference method. The models enable student to experiment with the inputs and investigate the behavior of the systems.

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The paper gives three examples of numerical modeling with spreadsheets in secondary school (age 15+) physics and biology − free fall in the air, animal population, and damped oscillation. The aim is to introduce the reader to numerical modeling. The simplest numerical methods are used − Euler's, and the finite difference method. The models enable student to experiment with the inputs and investigate the behavior of the systems.

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1. Introduction

During secondary school study, students are acquainted with various dynamic systems. The behavior of these can be described by differential equations. However, calculus is usually taught in the highest grade, so the systems are studied in a simplified way, and the conditions are idealized. It is rather frustrating then for a young scientist to hear at the end of a physics lesson that all the presented theory is a fairy tale as the main precondition is usually not fulfilled (vacuum in the free fall of a body), or even cannot be fulfilled (eliminating the damping force of a harmonic oscillator). In some cases, the precondition can be fulfilled easily (amplitude smaller than 5° degrees of a simple pendulum); in some cases, there is no idealization (animal population). Hence, some standard questions arise among students who are genuinely interested in the matter: what is the solution, or what is the behavior of the real system without the idealization?

One way to answer the question, and to enlarge the knowledge of passionate young scientists, is by making models using numerical methods. The mathematics of those is easy to comprehend. The main precondition for getting a model of sufficient precision is to divide the interval of the independent variable into a large number of subintervals. As the length of these approaches zero, the ratio of finite differences approaches the derivative, e.g. $v = \lim_{\Delta t \to 0}(\Delta s/\Delta t) = \mathrm{d}s/\mathrm{d}t$. A spreadsheet program is the ideal tool to carry out such a model. It presents an interactive form that enables students to experiment and find boundary or limiting cases.

In the paper, three examples of such modeling in secondary school physics and biology are given − free fall in the air, damped oscillation, and animal population. They were made in Excel 2003. The only skill that is required is making the maximum ranges of the axes in the charts constant, if necessary.

2. Free fall in the air

The acceleration of free fall g is commonly measured in secondary school physics. One method is based on the free fall of a solid body of constant mass m from a known height while a stopwatch measures the duration of the fall. If the fall takes place in a vacuum, then only the gravity force $G = gm$ acts on the body (downwards). Let the positive y semi-axis be oriented downwards, and let the body start at time t = 0 from position y = 0 at velocity v = 0. The equation of motion is (we assume constant mass m)

$$m\frac{\mathrm{d}v}{\mathrm{d}t} = mg\,,\ v(0) = 0\,,\ y(0) = 0\,, \tag{1}$$

or, as approximated by using a discrete difference equation in the first grade of secondary school (age 15 in Slovakia; four grades altogether),

$$m\frac{\Delta v}{\Delta t} = mg\,,\ v(0) = 0\,,\ y(0) = 0\,. \tag{2}$$

The velocity v and position y of the body is given by the well-known formulas [1]

$$v = gt\,,\ y = \frac{1}{2}gt^2\,. \tag{3}$$

The graphs of v and y against t are in Fig. 1.

Figure 1: Velocity and distance of a free falling ball in the air

These formulas are only valid in a vacuum. We mark them y_v and v_v. However, the fall usually takes place in the air. There are two forces that act on a falling body – gravity

$G = gm$ (downwards) and the air resistance force (drag) $F = CS\rho_a v^2 / 2$ (upwards) [2], where C is the drag coefficient that is dependent on the shape of the body, S is the maximum cross-section area of the body perpendicular to the direction of the velocity, and ρ_a is the density of the air. We assume the body to be solid enough not to lose weight due to the drag and temperature rise, i.e. we assume constant mass m, again. The equation of motion is

$$m\frac{\mathrm{d}v}{\mathrm{d}t} = mg - \frac{1}{2}CS\rho_a v^2,\ v(0) = 0,\ y(0) = 0. \tag{4}$$

or

$$\frac{\mathrm{d}v}{\mathrm{d}t} = g - Av^2,\ v(0) = 0,\ y(0) = 0, \tag{5}$$

or, as approximated by using a discrete difference equation,

$$\frac{\Delta v}{\Delta t} \approx g - Av^2,\ v(0) = 0,\ y(0) = 0, \tag{6}$$

where

$$A = \frac{CS\rho_a}{2m}. \tag{7}$$

We assume that the body is a ball with diameter d and density ρ_b. Then,

$$A = \frac{3}{4}\frac{C}{d}\frac{\rho_a}{\rho_b}. \tag{8}$$

Parameters C, d, ρ_a, ρ_b, g are the inputs. For a ball, C ranges from 0.1 to 0.5 depending on the smoothness of the surface [3]. If $A = 0$, we get the case in which the task and the solution are in a vacuum.

Suppose that the fall takes place in time interval $\langle 0, t_{max} \rangle$. We divide the interval into n equal segments of length $\Delta t = t_i - t_{i-1}$. We get time points t_i, $i = 0, 1, \cdots, n$, where $t_n = t_{max}$ and

$$t_i = t_{i-1} + \Delta t,\ t_0 = 0. \tag{9}$$

The velocity increment at time t_i results from equation (6)

$$\Delta v_i = \left(g - Av_i^2\right)\Delta t. \tag{10}$$

The velocity at time t_i is

$$v_i = v_{i-1} + \Delta v_{i-1}, \tag{11}$$

or, using equation (10) at index (*i*-1)

$$v_i = v_{i-1} + \left(g - Av_{i-1}^2\right)\Delta t\,,\; i = 1, \cdots, n\,,\; v_0 = 0\,. \tag{12}$$

Recurrence equations (9), (12) enable one to compute points (t_i, v_i) of the velocity graph. The result for $n = 1000$ is in Fig. 1. The gray cells contain formulas. The columns for i, t_i, v_{v}, v_i, y_{v}, y_i continue downwards for 1000 rows (hidden columns V − AG).
Since

$$v = \frac{\mathrm{d}y}{\mathrm{d}t} \approx \frac{\Delta y}{\Delta t}\,, \tag{13}$$

then

$$\Delta y_i = v_i\,\Delta t\,. \tag{14}$$

As

$$y_i = y_{i-1} + \Delta y_{i-1}\,, \tag{15}$$

we get

$$y_i = y_{i-1} + v_{i-1}\,\Delta t\,,\; y_0 = 0\,. \tag{16}$$

Recurrence equations (9), (16) enable one to compute points (t_i, y_i) of the distance graph. The result at $n = 1000$ is in Fig. 1.

As *G* is constant, and the resistance force *F* increases with the ball's velocity, the latter has to equalize the first one at some time. From then on, the body continues moving with constant velocity $v_{\max}$ − the terminal velocity of the fall. The equation *G* = *F* yields $v_{\max} = \sqrt{g/A}$. The graph of this constant function is a horizontal line − an asymptote to the velocity graph (see Fig. 1). We create it by designing a two-point graph given by points $(0, v_{\max})$ and $(t_{\max}, v_{\max})$.

The distance graph of a uniformly moving body is a line. Consequently, when the ball's velocity approaches the asymptotic velocity $v_{\max}$, the distance graph has to merge with a line − the asymptote to the graph. The distance between the points of the graph and the asymptote decreases as time *t* increases. Then, at any accuracy that one works with, there has to be a time point from which on the distance is too small to be considerable, and we can omit that i.e. we assume the next points of the graph to be the points of the asymptote. Hence, if $t_{\max}$ is big enough (see below), we may take points (t_n, y_n), (t_{n-1}, y_{n-1}) for the points of the asymptote and get the equation of the asymptote, which is

$$y = \frac{y_n - y_{n-1}}{t_n - t_{n-1}}\left(t - t_n\right) + y_n = \frac{y_n - y_{n-1}}{\Delta t}\left(t - t_n\right) + y_n = v_{n-1}\left(t - t_n\right) + y_n\,, \tag{17}$$

where we use equation (16). As $v_{n-1} = v_{\max} = \sqrt{g/A}$ (see above), the asymptote is

$$y = \sqrt{g/A}\left(t - t_n\right) + y_n \,. \tag{18}$$

We use the following procedure to get the asymptote: whenever we change inputs *C*, *d*, ρ_{a}, ρ_{b}, *g*, we adjust $t_{\max}$ to a value big enough to get a velocity graph with clear asymptotical part (it is $t_{\max} = 30$ sec in Fig. 1). The values of t_n, y_n are computed in the last row (1007) of the table. We refer to those cells in the cells labeled "help" (hidden) to get the values at our disposal. Then, we copy t_n, y_n to the clipboard, and paste special the values (only) into the white cells labeled "asymptote" (the asymptote is made as a two-point graph upon these four cells). Then, we adjust $t_{\max}$ to the previous value (13 sec in this case). The graphs change back, but the asymptote stays steady. A short macro can execute the copying and pasting (see button "asymptote" in Fig. 1), however, there is no possibility to undo the previous actions.

The analytic solution to equation (5) is [2, 4]

$$v = \sqrt{\frac{g}{A}} \tanh\left(t\sqrt{gA}\right),\; y = \frac{1}{A}\ln\left[\cosh\left(t\sqrt{gA}\right)\right]. \tag{19}$$

For the chosen ball (diameter *d* = 2 cm, made of iron $\rho_{\mathrm{b}} = 7800$ kgm^{-3}, drag coefficient *C* = 0.4, air density $\rho_{\mathrm{a}} = 1.29$ kgm^{-3}, $t_{\max} = 13$ sec divided into 1000 segments, gravity acceleration $g = 9.81$ ms^{-2}), the maximum error in velocity is -0.05 % at *t* = 5.42 sec; the error at *t* = 13 sec is -0.02 %. The error in distance is 100 % at *i* = 1 (*t* = 0.013 sec) and decreases rapidly to 10 % at *i* = 10 (*t* = 0.13 sec), to 1 % at *i* = 98 (*t* = 1.28 sec), and to 0.1 % at *i* = 587 (*t* = 7.63 sec); the error at *t* = 13 sec is 0.04 %.

Remarks: For numerical ordinary differential equations see [5]; for spreadsheet analysis of school experiments with falling balls with quadratic drag see [6]; for modeling projectile motion with quadratic drag with spreadsheets see [7]; for accuracy in computing acceleration of free fall if equation (3) is used instead of equation (19) see [8].

3. Model of animal population

In the logistic growth model, the number *N* of animals of one species that live in a closed territory and feed on food only produced there is given by differential equation [9, 10]

$$\frac{\mathrm{d}N}{\mathrm{d}t} = rN - \frac{r}{K}N^2,\; N(0) = N_0\,, \tag{20}$$

or, approximately,

$$\frac{\Delta N}{\Delta t} = rN - \frac{r}{K}N^2,\; N(0) = N_0\,, \tag{21}$$

where *r* is the growth rate (increment in number of specimen per head and time unit; it is 5.07; 0.39; 0.23; 0.0125 [11] per year for the field mouse, sardine, roe deer, and man worldwide, respectively), *K* is the carrying capacity of the habitat (maximum number of specimen that the territory can provide with food), and N_0 is the number of individuals at the start. Quantities *r*, *K*, N_0 are the inputs.

Suppose that the system develops over the time interval $\langle 0, t_{\max} \rangle$. We divide the interval into *n* equal segments of length $\Delta t = t_i - t_{i-1}$. We get time points t_i, $i = 0, 1, \cdots, n$, where $t_n = t_{\max}$ and

$$t_i = t_{i-1} + \Delta t,\ t_0 = 0. \tag{22}$$

The incremental growth of the number *N* at time t_i is

$$\Delta N_i = \left(rN_i - \frac{r}{K} N_i^2 \right) \Delta t. \tag{23}$$

The number of animals in time t_i is then

$$N_i = N_{i-1} + \Delta N_{i-1},\ i = 1, \cdots, n,\ N_0 \text{ is given} \tag{24}$$

or

$$N_i = N_{i-1} + \left(rN_{i-1} - \frac{r}{K} N_{i-1}^2 \right) \Delta t,\ i = 1, \cdots, n. \tag{25}$$

Recurrence equations (22), (25) enable one to compute points (t_i, N_i) of the graph. The result at $n = 1000$ is in Fig. 2. The gray cells contain formulas. The columns for i, t_i, and N_i, continue downwards for 1000 rows (hidden columns K – M).

If $t_{\max} >> 1$, the number *N* approaches the capacity *K* (*N* cannot exceed *K*). Hence, the line $N = K$ is an asymptote of the graph. We can create it as a two-point graph given by points $(0, K)$ and $(t_{\max}, K)$.

The analytic solution to equation (20) is [10]

$$y = \frac{y_0}{y_0 + (1 - y_0)\mathrm{e}^{-rt}}, \tag{26}$$

where $y = N/K$, $y_0 = N_0/K$; $y, y_0 < 1$.

For the chosen system (growth rate *r* = 0.39, capacity *K* = 100, starting number $N_0 = 2$, $t_{\max} = 30$ years divided into 1000 segments), the maximum error in *N* is 0.98 % at t = 7 years .

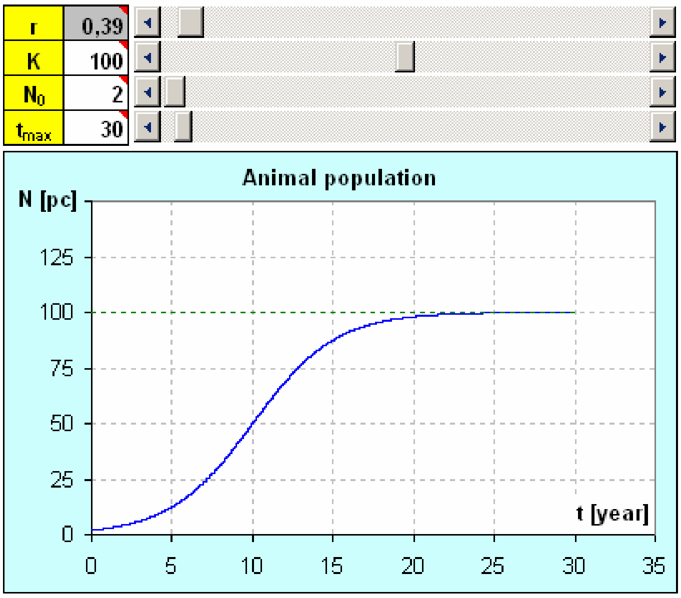

Figure 2: Population of animals in a closed territory

4. Damped oscillation of a body on spring

A body on spring is a subject to two forces – the elasticity of the spring, and the gravity. The resultant force is directly proportional to the deflection y and acts against it. If the body starts at time t = 0 from position y = A (A is the amplitude) at velocity v = 0, then the equation of motion is

$$m\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} = -ky\,,\ y(0) = A\,,\ v(0) = 0\,. \tag{27}$$

or

$$ma = -ky\,,\ y(0) = A\,,\ v(0) = 0\,, \tag{28}$$

where k is the elastic constant of the spring, and a is the acceleration.
The solution is [12, 13]

$$y = A\cos(\omega t)\,, \tag{29}$$

where $\omega = \sqrt{k/m}$. (We note that the derivation of equation (29) in the first grade of secondary school is often based on a parallelism between the oscillator and circular motion, in which case the method is very vague.)
equation (29) describes the motion of an ideal oscillator that moves without any friction. However, the friction force always acts between the particles of a real spring, and that dampen the oscillation until it stops (moreover, there is the air drag, but we can eliminate that if we put the oscillator in a vacuum chamber). The friction force is directly proportional to the velocity v, and acts against the deflection y. The equation of motion is

$$m\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} = -rv - ky\,, \tag{30}$$

where r is the damping coefficient. Introducing $2\alpha = r/m$, we get

$$\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} + 2\alpha\frac{\mathrm{d}y}{\mathrm{d}t} + \omega^2 y = 0\,,\ y(0) = A\,,\ v(0) = 0\,. \tag{31}$$

Suppose that the oscillation take place in time interval $\langle 0, t_{\max}\rangle$. We divide the interval into n equal segments of length $\Delta t = t_i - t_{i-1}$. We get time points t_i, $i = 0, 1, \cdots, n$, where $t_n = t_{\max}$ and

$$t_i = t_{i-1} + \Delta t\,,\ t_0 = 0\,. \tag{32}$$

We approximate [14]

$$\frac{\mathrm{d}y}{\mathrm{d}t} \approx \frac{\Delta y}{\Delta t} = \frac{y_i - y_{i-1}}{\Delta t}\,, \tag{33}$$

$$\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} = \frac{\mathrm{d}\left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)}{\mathrm{d}t} \approx \frac{\Delta\left(\frac{\Delta y}{\Delta t}\right)}{\Delta t} = \frac{\frac{y_{i+1} - y_i}{\Delta t} - \frac{y_i - y_{i-1}}{\Delta t}}{\Delta t} = \frac{y_{i+1} - 2y_i + y_{i-1}}{(\Delta t)^2}\,, \tag{34}$$

$$y(0) = y_0 = A\,,\ v(0) = \left.\frac{\Delta y}{\Delta t}\right|_{t=0} = \frac{y_1 - y_0}{\Delta t} = 0\,. \tag{35}$$

Substitution of equation (33 − 35) into equation (31) gives

$$y_{i+1} = \left(2 - 2\alpha h - \omega^2 h^2\right) y_i + (2\alpha h - 1) y_{i-1}\,,\ y_0 = A\,,\ y_1 = A\,, \tag{36}$$

where h is for Δt to simplify the notation.
Recurrence equations (32), (36) enable one to compute points (t_i, y_i) of the graph. The result for $n = 1000$ is in Fig. 3. The gray cells contain formulas. The columns for i, t_i, y_i continue downwards for 1000 rows (hidden columns K − M).

If we put $r = 0$, then we get the free oscillation. If we add the analytic graph given by equation (29) into the chart, then we can check that the graphs are the same (this is a demonstration that equation (29) is valid − the validity of such a "proof" is comparable to the validity of the above-mentioned parallelism; the maximum difference between the numerical and the analytic model is ± 0.024).

The analytic solution to equation (31) is [12, 13]

$$\alpha > \omega:\quad y = A\,\mathrm{e}^{-\alpha t}\left[\cosh(t\sqrt{\alpha^2 - \omega^2}) + \alpha\frac{\sinh(t\sqrt{\alpha^2 - \omega^2})}{\sqrt{\alpha^2 - \omega^2}}\right] \tag{37}$$

$$\alpha = \omega:\quad y = A\,\mathrm{e}^{-\alpha t}(1 + \alpha t) \tag{38}$$

$$\alpha < \omega: \quad y = A\,\mathrm{e}^{-\alpha t}\left[\cos(t\sqrt{\omega^2 - \alpha^2}) + \alpha\,\frac{\sin(t\sqrt{\omega^2 - \alpha^2})}{\sqrt{\omega^2 - \alpha^2}}\right] \tag{39}$$

The error of the numerical model at the stated inputs ($A = 2$, $m = 0.1$, $r = 0.02$, $k = 0.14$, and $t_{max} = 20$ sec divided to 1000 segments) is 0.885 % at $t = 16$ sec.

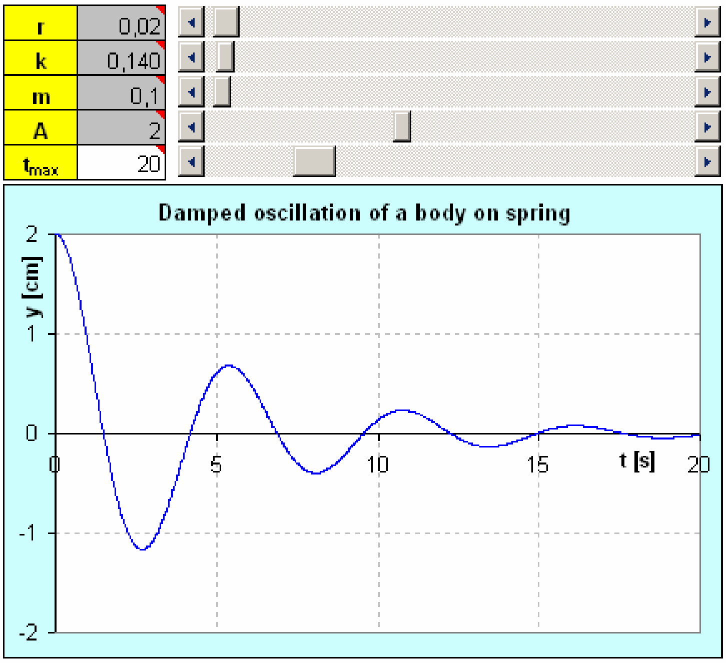

Figure 3: Damped oscillation of a body on spring

5. Conclusion

The examples show that it is possible to model dynamic systems without the knowledge of the differential equations that describe them, as well as the analytic solutions. The difference equations, which are presented to students in secondary school, give a good approximation even in the case of the second derivative, and the numerical models are precise enough compared to the analytic solution.

The aim of the paper is to introduce the reader to numerical modeling. That is why we have used the simplest numerical methods − Euler's method in the first and the second example, and the finite difference method in the third one. We note that increasing the number of divisional points of the main time interval e.g. to 5000 makes the methods considerably more precise. There are more precise methods that might have been used, but they are more complicated. The interested persons can find them in the literature.

The models mirror well physical reality. They enable student to experiment with the inputs and look for relations between them and the behavior of the system. The models are also useful for showing and studying the limiting cases that exist at each system.

We refer to Wikipedia on purpose to get the information at students' disposal.

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