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This paper reveals a number of learning activities emerging from a spreadsheet-generated multiplication table. These activities are made possible by using such features of the software as conditional formatting, circular referencing, calculation through iteration, scroll bars, and graphing. The paper is a reflection on a mathematics content course designed for prospective elementary teachers using the hidden mathematics curriculum framework. It is written in support of standards for teaching and recommendations for teachers in North America.

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1 Introduction

This paper reflects on the author's experience in teaching a technology-enhanced course in mathematics content to prospective elementary teachers (referred to below as teachers) using the hidden mathematics curriculum framework [3]. This (positive) learning framework is based on the notion that many seemingly routine mathematical tasks, when explored beyond the boundaries of the traditional elementary curriculum, can be used as windows on big ideas obscured in the curriculum due to the expected simplicity of the tasks' didactical orientation on the development of basic skills. Also, many seemingly disconnected mathematical problems and ideas scattered across the whole K-12 mathematics curriculum, in reality, share common conceptual structure hidden from learners because of its intrinsic complexity. Technology has great potential to support such explorations and to relate different problems and ideas across the curriculum.

The current standards for teaching mathematics at the elementary level [9], [10] as well as recommendations for elementary teacher preparation programs [6] focus on concept oriented problem-solving activities. In order to be able to teach mathematics

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conceptually, teachers themselves have to be taught in that way. It has been suggested elsewhere [2] that using the hidden mathematics curriculum framework in the technological paradigm, in particular, using visualization features of an electronic spreadsheet, provides significant opportunities to enhance conceptual component of mathematics teacher education. By integrating spreadsheet modeling and appropriate pedagogical mediation, many complex mathematical ideas can become embedded in this powerful software tool enabling the teachers to understand these ideas.

Such mediation occurs in the social context of competent guidance provided by the instructor who serves the teachers as 'a more knowledgeable other' [13]. The transactional nature of learning, being the primary thesis of Vygotskian educational psychology, is also reflected in Freudenthal's [7] interpretation of the didactic phenomenology of mathematics – "a way to show the teacher the places where the learner might step into the learning process of mankind" (p. ix). This combination of Vygotskian tradition to see teaching as the transfer of knowledge and Freudenthal's perspective on mathematics learning as the advancement of the culture of mankind provides a theoretical foundation for the hidden mathematics curriculum framework used in the context of teacher education.

One of three major themes within mathematics curriculum for grades 3-5 in North America [9] is the multiplicative reasoning. The basic topic here is the multiplication of whole numbers. Typically, the study of this topic converges to the memorization of basic multiplication facts presented in the form of what is commonly referred to as the multiplication table. Many prospective elementary teachers enter teacher education programs having a very traditional (and often frustrating) experience with the table as they recall long hours spent on memorizing the table without paying any attention to its rich conceptual structure. The Conference Board of the Mathematical Sciences [6] argued for the importance of learning environments for the teachers that enable them "to create meaning for what many had only committed to memory but never really understood" (p. 18). The learning environment of the multiplication table, when viewed through the lens of the hidden mathematics curriculum framework, represents a large source of conceptually rich activities that can be used to enhance a conceptual component in the teaching of school mathematics.

These activities can be supported by the use of a number of advanced features of a spreadsheet. Indeed, the software makes it possible to construct dynamic leaning environments conducive to the discovery of many interesting properties of integers hidden in a static multiplication table. In what follows, using Steffe's [12] terminology, such environments will be referred to as "possible learning environments." Steffe introduced this didactical construct to suggest that in the design of instructional activities one should always look for a possibility of going beyond the traditional expectations for students' learning. This approach can be extended by developing possible learning environments within the hidden mathematics curriculum framework. Below the term possible learning environment (PLE) will be used in that extended sense. Using the multiplication table as a background, it will be demonstrated how a number of PLEs made possible by the use of a spreadsheet can facilitate informed entries into "the learning process of mankind" for the teachers.

2 The multiplication table as a spreadsheet

Structurally, the multiplication table consists of the products of two positive integers (inputs of the table) each of which varies over a given range. This seemingly mundane array of numbers can be used for the introduction of many concepts taught at the elementary level and beyond it. First of all, by using the table, teachers can be reminded about such basic concepts as commutative law of multiplication, distributive law of multiplication over addition, factorization, divisibility, and symmetry. Furthermore, there are many interesting relationships among positive integers implicitly present in the multiplication table. These relationships can be revealed through the combination of computer activities and follow-up tasks that are reflections on these activities. Such an approach opens the gates into a number of basic concepts of algebra and discrete mathematics that teachers should be familiar with. More specifically, the concepts of triangular numbers, summation formulas, counting techniques, and mathematical induction proof can come into play through exploring numbers in the multiplication table.

SIZE 10

	1	2	3	4	5	6	7	8	9	10
1	1	2	3	4	5	6	7	8	9	10
2	2	4	6	8	10	12	14	16	18	20
3	3	6	9	12	15	18	21	24	27	30
4	4	8	12	16	20	24	28	32	36	40
5	5	10	15	20	25	30	35	40	45	50
6	6	12	18	24	30	36	42	48	54	60
7	7	14	21	28	35	42	49	56	63	70
8	8	16	24	32	40	48	56	64	72	80
9	9	18	27	36	45	54	63	72	81	90
10	10	20	30	40	50	60	70	80	90	100

Figure 1: 10×10 multiplication table.

As shown in the spreadsheets pictured in Figures 1 and 2, column `A` and row 4 are filled with positive integers beginning from 1 (for which the names `x` and `y`, respectively, will be used throughout the paper). This range can be made variable; i.e., dependent on the size of the table[1]. To this end, one can enter cells `B4` and `A5` with the unity and, by incorporating simple inequalities, define the following formulas in cells `C4`

[1] Note that syntactic versatility of spreadsheets enables one to construct both visually and computationally identical environments using syntactically different formulas. For example, one can use the feature of not showing zeros in active cells when constructing multiplication tables with variable ranges. The author's choice of an alternative syntax in setting cells blank is due to the possibility of extending the tables and associated activities to modular arithmetic [4] where products of non-zero factors may have zero values.

and A6, respectively: `=IF(B4<n,B4+1," ")`, `=IF(A5<n,1+A5," ")`. In these formulas (replicated across row 4 and down column A), the name `n` stands for the size of the multiplication table and varies in a slider-controlled cell D2. Entering cell B5 with the spreadsheet formula `=IF(OR(x=" ", y=" ")," ", x*y)` and replicating it across the columns and down the rows results in the array of numbers known as the multiplication table.

SIZE 12

	1	2	3	4	5	6	7	8	9	10	11	12
1	1	2	3	4	5	6	7	8	9	10	11	12
2	2	4	6	8	10	12	14	16	18	20	22	24
3	3	6	9	12	15	18	21	24	27	30	33	36
4	4	8	12	16	20	24	28	32	36	40	44	48
5	5	10	15	20	25	30	35	40	45	50	55	60
6	6	12	18	24	30	36	42	48	54	60	66	72
7	7	14	21	28	35	42	49	56	63	70	77	84
8	8	16	24	32	40	48	56	64	72	80	88	96
9	9	18	27	36	45	54	63	72	81	90	99	108
10	10	20	30	40	50	60	70	80	90	100	110	120
11	11	22	33	44	55	66	77	88	99	110	121	132
12	12	24	36	48	60	72	84	96	108	120	132	144

Figure 2: 12×12 multiplication table.

By changing the value of the slider-controlled cell D2, one can explore and interpret the meaning of the appearance of certain integers in a table that were absent in a smaller one. For example, teachers can be asked to explain as to why the number 96 being present in the 12×12 multiplication table (Figure 2), is absent in the 10×10 table (Figure 1). In that way, teachers can develop an appreciation of reflective inquiry pedagogy, which includes not only a skill to observe a certain phenomenon, but also an ability to explain it.

3 PLE 1: Conditional formatting as a practice in developing mathematical definition

Conditional formatting is a modern spreadsheet capability that allows for automatic formatting of any cell based on its current value [4]. However, when a range of cells, each of which has a different numerical value, has to be uniformly formatted, it is not a specific value that defines formatting but rather a general rule that describes numbers in this range. Consider the multiplication table of Figure 1. One can divide all numbers in the table into three groups: numbers on a diagonal (bottom left–top right or top left–bottom right), numbers below the diagonal, and numbers above it. Figures 3 and 4 show the three groups of numbers formatted differently. How can this be done?

With this in mind, one can note that each product that does not belong to the bottom left-top right diagonal consists of factors the sum of which is either greater or smaller than the size of the table plus 1. Thus the table shown in Figure 3 can be formatted using the following conditions:

Condition 1. Formula is `=B$4 + $A5 < n + 1`; choose format.

Condition 2. Formula is `=B$4 + $A5 > n + 1`; choose format.

On the other hand, each product that resides above the main (top left – bottom right) diagonal has the first factor greater that the second factor. The inequality between the two factors changes if one considers products that reside below the main diagonal. Thus the table shown in Figure 4 can be formatted using the following conditions:

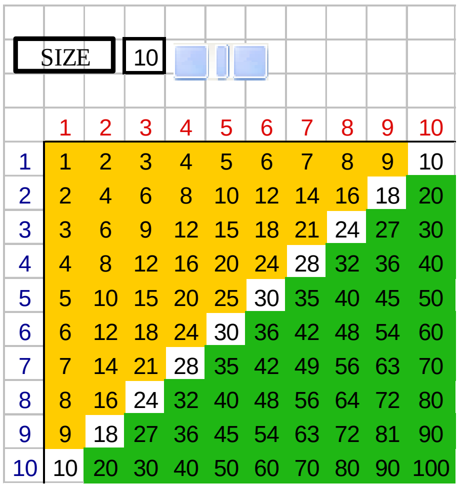
Condition 1. Formula is `=AND(B$4 > $A5, B$4 <= n, $A5 <= n)`; choose format.

Condition 2. Formula is `=AND(B$4 < $A5, B$4 <= n, $A5 <= n)`; choose format.

Note that in the case of Figure 4, Conditions 1 and 2 are more complicated than those used in formatting the spreadsheet of Figure 3. The point is that in order for a chosen format to be dependent on the size of the table, the variable n should be incorporated into formulas responsible for conditional formatting. So, whereas in the case of Figure 3 the inequalities linking the inputs of the table include the name n, in the case of Figure 4 one has to artificially use this name in the conditional formatting formulas. This explains the need to use a logical operator `AND` in order to show the symmetrical split through conditional formatting. In that way, the use of this modern spreadsheet capability can become a practice for teachers in understanding and demonstrating as to how "software can embody a mathematical definition" [6, p132]. Surprisingly, the definitions allowing for different splits of contents in the table are based on simple inequalities that not only demonstrate their mathematical power, but also serve to reinforce one's understanding of the meaning of numbers in the multiplication table.

4 PLE 2: Representing the multiplication table as an iterative structure

The multiplication table provides teachers with an environment through which algebraic notation – "an efficient means for representing properties of operations and relations among them" [6, p20] – can come into play. Furthermore, patterns observed in the multiplication table can be interpreted in the language of discrete mathematics; indeed, counting numbers – the building blocks of the table – are discrete concepts. Toward this end, one can observe that each number in the table beginning from its second column (column C in the spreadsheets of Figures 1-4) equals to the number immediately to the left plus the number in the first column of the table (column B in Figures 1-4). Alternatively, each number in the table beginning from its second row (row 6 in Figures 1-4) equals to the number immediately above plus the number in the first row. Teachers can be encouraged to interpret these observations by connecting multiplication



SIZE 10

	1	2	3	4	5	6	7	8	9	10
1	1	2	3	4	5	6	7	8	9	10
2	2	4	6	8	10	12	14	16	18	20
3	3	6	9	12	15	18	21	24	27	30
4	4	8	12	16	20	24	28	32	36	40
5	5	10	15	20	25	30	35	40	45	50
6	6	12	18	24	30	36	42	48	54	60
7	7	14	21	28	35	42	49	56	63	70
8	8	16	24	32	40	48	56	64	72	80
9	9	18	27	36	45	54	63	72	81	90
10	10	20	30	40	50	60	70	80	90	100

Figure 3: A non-symmetrical split.

to repeated addition; indeed, each number in the table, being a product of two factors, develops by iterating one of its factors. Algebraically, this can be represented through the identities $xy = (x-1)y+y$ and $xy = (y-1)x+x$, respectively. In a more formalized notation, setting $P(x,y) = xy$ these identities can be expressed through what may be referred to as partial difference equations in two variables [8]

$$P(x,y) = P(x-1,y) + y \tag{1}$$

or

$$P(x,y) = P(x,y-1) + x \tag{2}$$

satisfying boundary conditions $P(1,y) = y$ and $P(x,1) = x$, respectively.

Teachers can use either definition (1) or definition (2) in order to generate the multiplication table through this iterative approach as an alternative to a straightforward multiplication. Furthermore, a number of simple identities can be formulated, proved, and then interpreted in terms of the properties of operations involved:

$$\begin{aligned}
P(x,y+m) &= P(x,y) + my \\
P(x+m,y) &= P(x,y) + mx \\
P(x,y+m) &= P(x,y) + P(x,m) \\
P(x+m,y) &= P(x,y) + P(m,y)
\end{aligned}$$

In that way one can be shown how, at a higher level of generalization, simple concepts could be represented through the more complicated ones. In other words, a topic of discrete mathematics – recurrences expressed by partial difference equations in two variables – belongs to the hidden curriculum of the elementary school mathematics.

SIZE	10									
	1	2	3	4	5	6	7	8	9	10
1	1	2	3	4	5	6	7	8	9	10
2	2	4	6	8	10	12	14	16	18	20
3	3	6	9	12	15	18	21	24	27	30
4	4	8	12	16	20	24	28	32	36	40
5	5	10	15	20	25	30	35	40	45	50
6	6	12	18	24	30	36	42	48	54	60
7	7	14	21	28	35	42	49	56	63	70
8	8	16	24	32	40	48	56	64	72	80
9	9	18	27	36	45	54	63	72	81	90
10	10	20	30	40	50	60	70	80	90	100

Figure 4: A symmetrical split.

5 PLE 3: The sum of all numbers in the multiplication table

An interesting inquiry into the properties of numbers in the $n \times n$ multiplication table deals with finding the sum of all numbers in this table as a function of n. As shown in [9], the total number of rectangles within the $n \times n$ checkerboard can be found by adding up all numbers in the corresponding multiplication table of the same size. This hidden connection between the checkerboard and the multiplication table; that is, connection between geometric and arithmetic structures, can be revealed first by exploring the dynamic multiplication table for small values of n and then, a counting technique so discovered, can be generalized to the table of any size.

The sum of all numbers in the $n \times n$ multiplication table, for which the notation $S(n)$ will be used below, can be first found computationally for specific values of n. To begin, note that the formula `=SUM(B5:K14)` defined in cell `P2` when applied to the 10×10 multiplication table of Figure 5 returns the number 3025. One may wonder: What is the connection between this number and the size of the multiplication table? In other words, how are the numbers 3025 and 10 related? In order to explore these questions, one can use the multiplication table of a variable size and construct a tabular representation of the function $S(n)$.

A powerful feature of a spreadsheet is its capability of interactive dynamic construction of a verbally defined function. The basic computational idea enabling such a construction is the use of a circular reference in a spreadsheet formula; i.e., a reference to a cell in which the formula is defined [1]. More specifically, in the spreadsheet of Figure 5 cell `P5` is entered with the formula `=IF(n=0," ",IF(n=O5,sum,P5))` which is then replicated down column `S`. The formula refers to the size of the table and the sum of numbers in it through, respectively, the names `n` and `sum`. It includes a circular reference, P5, which can be resolved by choosing iteration as the preferred way of calculation (Preferences on the Mac or Tools | Options on the PC). This makes it possible for a spreadsheet to keep an already computed value of $S(n)$ unchanged as the content of cell `P2` (that is, sum) changes. The two-column representation of the function $S(n)$ for $1 \leq n \leq 10$ can be used to guess the formula for $S(n)$ prior to its formal demonstration. It appears that all values of $S(n)$ are square numbers. Furthermore, teachers can check to see that these are squares of the numbers $1, 3, 6, 10, 15, 21, 28, 36, 45$, and 55. Surprise! The ten listed numbers are triangular numbers of ranks equal to the size of a corresponding multiplication table. This completes "the inductive phase" [11, p108] of finding the sum of all numbers in the $n \times n$ multiplication table; i.e., the following technology-motivated formula can be conjectured

$$S\left(n\right) = \left(\frac{n\left(n+1\right)}{2}\right)^2 \tag{3}$$

Formula (3) can be proved by the method of mathematical induction referred to by Polya [11] as "the demonstrative phase" (p. 110) or, alternatively, "passing from n to $n + 1$" (p. 111). The use of a spreadsheet made it possible to computationally enhance the inductive phase. Furthermore, as will be shown below, the conditional formatting feature of the software allows for the demonstrative phase to be visually enhanced.

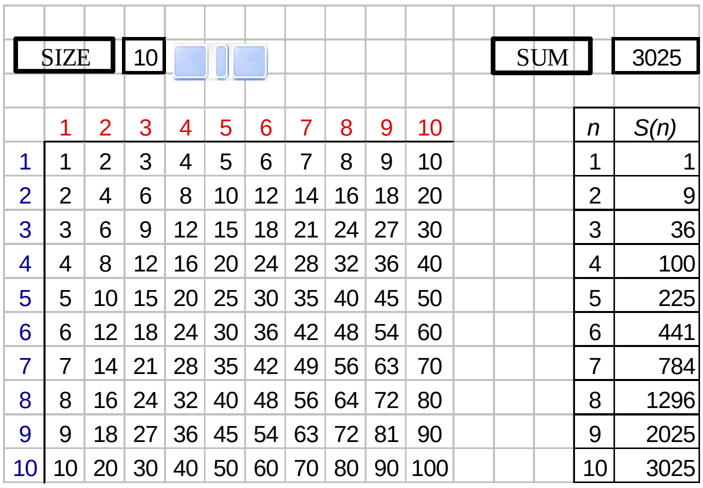
The first step of the latter phase is to show that formula (3) is true for $n = 1$. Indeed, when $n = 1$, the multiplication table consists of the unity only whence $S\left(1\right) = 1$. The same value has the right-hand side of formula (3) when $n = 1$. One should not underestimate the didactical importance of this elementary demonstration for it establishes an important link between the concreteness of the multiplication table and the abstractness of formula (3).

The second step of the demonstrative phase is to "test the transition from n to $n+1$" [11, p112]. Assuming that formula (3) holds true for n (in other words, making what can be referred to as inductive assumption), one has to show that after replacing n by $n + 1$ it remains true; that is:

$$S\left(n+1\right) = \left(\frac{\left(n+1\right)\left(n+2\right)}{2}\right)^2$$

To visually support the second step; in other words, in order to show what augments the table when it acquires new row and new column, one can use conditional formatting. This feature can highlight any augmentation of the table from its current state by a gnomon. As shown in Figure 6, conditional formatting highlights such a gnomon consisting of numbers that belong to $S(11)$ but do not belong to $S(10)$. More specifically, $S(11) - S(10) = 2 \cdot 11 \cdot (1 + 2 + \ldots + 11) - 11^2$. In order to highlight numbers that emerge from the inductive transfer, the following formula can be entered into a conditional formatting dialogue box: Condition 1. Formula is `=AND(OR(B$4>n-1, $A5>n-1), COUNT(B4)>0, COUNT(A5)>0)`; choose format. Note $n = 11$ in Figure 6.

Hidden Mathematics of the Multiplication Table



SIZE 10 SUM 3025

	1	2	3	4	5	6	7	8	9	10
1	1	2	3	4	5	6	7	8	9	10
2	2	4	6	8	10	12	14	16	18	20
3	3	6	9	12	15	18	21	24	27	30
4	4	8	12	16	20	24	28	32	36	40
5	5	10	15	20	25	30	35	40	45	50
6	6	12	18	24	30	36	42	48	54	60
7	7	14	21	28	35	42	49	56	63	70
8	8	16	24	32	40	48	56	64	72	80
9	9	18	27	36	45	54	63	72	81	90
10	10	20	30	40	50	60	70	80	90	100

n	S(n)
1	1
2	9
3	36
4	100
5	225
6	441
7	784
8	1296
9	2025
10	3025

Figure 5: Finding $S(n)$ computationally.

In general, the transition from n to $n+1$ yields the relationship:

$$S(n+1) = S(n) + 2(n+1)(1+2+\ldots+(n+1)) - (n+1)^2$$

Next, by using the formula [11]

$$1+2+3+\ldots+n = \frac{n(n+1)}{2} \tag{4}$$

one can show that $S(n+1) = S(n) + (n+1)^3$. Finally, using the inductive assumption yields the following chain of equalities:

$$S(n+1) = S(n) + (n+1)^3 = \left(\frac{n(n+1)}{2}\right)^2 + (n+1)^3 = \left(\frac{(n+1)(n+2)}{2}\right)^2$$

This concludes the demonstrative phase of finding the sum of all numbers in the $n \times n$ multiplication table; i.e., a proof of formula (3) by the method of mathematical induction has been completed.

Note that one can derive formula (3) by using pure algebraic techniques. Indeed, in the $n \times n$ multiplication table, the sum $S(n)$ can be found by noting that the sum of numbers in each row of the table is a multiple of the sum of numbers in the first row. Then, using the distributive law of multiplication over addition and formula (4) yields

$$\begin{aligned} S(n) &= (1+2+3+\ldots+n) + 2(1+2+3+\ldots+n) + \ldots + n(1+2+3+\ldots+n) \\ &= (1+2+3+\ldots+n)^2 = \left(\frac{n(n+1)}{2}\right)^2 \end{aligned}$$

SIZE 11

	1	2	3	4	5	6	7	8	9	10	11
1	1	2	3	4	5	6	7	8	9	10	11
2	2	4	6	8	10	12	14	16	18	20	22
3	3	6	9	12	15	18	21	24	27	30	33
4	4	8	12	16	20	24	28	32	36	40	44
5	5	10	15	20	25	30	35	40	45	50	55
6	6	12	18	24	30	36	42	48	54	60	66
7	7	14	21	28	35	42	49	56	63	70	77
8	8	16	24	32	40	48	56	64	72	80	88
9	9	18	27	36	45	54	63	72	81	90	99
10	10	20	30	40	50	60	70	80	90	100	110
11	11	22	33	44	55	66	77	88	99	110	121

Figure 6: Conditional formatting illustrates inductive transfer.

6 PLE 4: Making mathematical connections

One can construct a simple spreadsheet to model numerically the sums of cubes of consecutive counting numbers starting from 1. Data so developed ("the inductive phase") enables one to conjecture and then prove by the method of mathematical induction ("the demonstrative phase") the following formula

$$1^3 + 2^3 + 3^3 + \ldots + n^3 = \left(\frac{n\,(n+1)}{2}\right)^2 \tag{5}$$

Comparing formulas (3) and (5) implies the fact that the sum of all numbers in the $n \times n$ multiplication table equals $1^3 + 2^3 + 3^3 + \ldots + n^3$. The latter can be shown to represent the sum of all cubes within a cube of side n. In other words, the number of rectangles within the $n \times n$ checkerboard equals the number of cubes within the $n \times n \times n$ cube. In order to explain this unexpected connection, all cubes in the $n \times n \times n$ cube can be arranged into n groups depending on the size of a cube. Then each such group can be mapped on a gnomon in one of the faces of the $n \times n \times n$ cube. By representing a face of the cube as the $n \times n$ multiplication table, one can show that the sum of all numbers in each gnomon equals to the number of cubes in the corresponding group. Indeed, the sum of numbers in the k-th gnomon, representing the total number of cubes of side $n - k + 1$, can be found as follows:

$$2\,(k + 2k + \ldots + (k-1)\,k) + k^2 = (k + k\,(k-1))\,(k-1) + k^2 = k^2\,(k-1) + k^2 = k^3$$

As has been shown above, the sum of numbers in all n gnomons equals to the total number of rectangles within the $n \times n$ face of the $n \times n \times n$ cube. In that way, geometric structures of different dimensions (i.e., cubes and rectangles) can become connected through the use of hidden mathematics of the multiplication table.

7 PLE 5: Percentage of squares among rectangles on the $n \times n$ checkerboard

It is interesting to note that finding the number of squares among all rectangles on the $n \times n$ checkerboard leads to the sum of squares of consecutive counting numbers starting from 1. The formula for this sum has the form (e.g., [11], [14])

$$1^2 + 2^2 + 3^2 + \ldots + n^2 = \frac{n\,(n+1)\,(2n+1)}{6} \tag{6}$$

The derivation of formula (6) is much more complex in comparison with that of formula (4) or formula (5) and is not discussed here. Teachers can use a spreadsheet to find the percentage of squares among rectangles on the $n \times n$ checkerboard for different values of n beginning from $n = 1$. Such a spreadsheet is shown in Figure 7. The chart embedded into it shows that percentage of squares among the rectangles decreases monotonically as n increases.

Note that simple programming of the spreadsheet of Figure 7 is based on formulas (3) and (6). It can be demonstrated formally that:

$$\frac{n(n+1)(2n+1)/6}{(n(n+1)/2)^2} = \frac{2(2+1/n)}{3(n+1)} \longrightarrow 0 \qquad \text{as} \quad n \longrightarrow \infty$$

In particular, teachers can be asked to find both computationally and algebraically the smallest value of n for which the percentage of squares among rectangles becomes smaller that 1%. It should be noted that algebraic part of this investigation might prove difficult for the teachers.

8 PLE 6: Counting numbers with special properties in the multiplication table

One can explore divisibility properties of numbers in the multiplication table. For example, one can find as to how many numbers in the 10×10 multiplication table are divisible by 2, 3, 4, etc., and then attempt to generalize the results to the $n \times n$ multiplication table. Figures 8 shows a spreadsheet enabling this kind of explorations. The spreadsheet formula `=IF(OR(x=" ", y=" ")," ", IF(MOD(x*y,MODULO)=0, x*y, " ",))` defined in cell `B5` and replicated across the columns and down the rows. In this formula, the name MODULO (defined in a slider-controlled cell `K2`) stands for a number, the multiples of which have been displayed by the spreadsheet. Cell `P2` contains the formula `=COUNT(B5:P19)` which counts the total number of such multiples. In particular, it turns out that there are 75 even products in the 10×10 table (Figure 8).

From this simple computational experiment a number of useful mathematical activities can emerge. It is not uncommon to hear from teachers that half of the products in the 10×10 multiplication table are even numbers. However, when confronted with the result of spreadsheet modeling, the meaning of the number 75 becomes clear to them.

n	R(n)	S(n)	S(n)/R(n)
1	1	1	100.0%
2	9	5	55.6%
3	36	14	38.9%
4	100	30	30.0%
5	225	55	24.4%
6	441	91	20.6%
7	784	140	17.9%
8	1296	204	15.7%
9	2025	285	14.1%
10	3025	385	12.7%
11	4356	506	11.6%
12	6084	650	10.7%
13	8281	819	9.9%

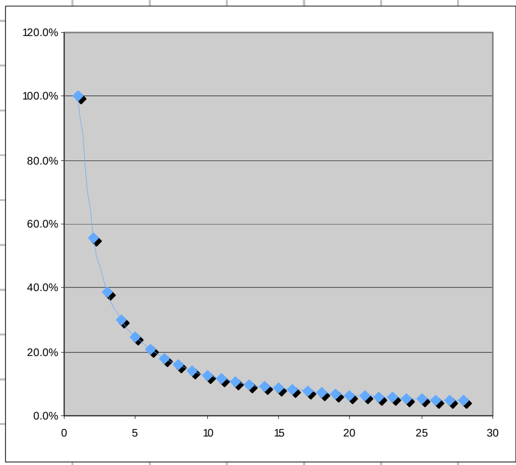


Figure 7: Modeling the percentage of squares among rectangles function.

Indeed, in the range 1 through 10 there are 5 odd numbers; thus there is 1 chance out of 4 to get an odd product when multiplying two numbers from this range. Consequently, there are 3 chances out of 4 to get an even product when multiplying two numbers from this range. In other words, 75% of products in the 10×10 multiplication table are divisible by 2.

In order to generalize this finding to the table of any size, one has to explore whether this percent changes as one alters the size of the table. To this end, one can change n (cell `D2`) from 10 to 9 and observe that in the 9×9 table (not shown here) there are 56 even numbers out of 81 total. This constitutes less than 70 % of all numbers in the table. Yet, in the 8×8 table (not shown here) there are 48 even numbers out of 64 total. Again, this is exactly 75%. These observations can motivate an algebraic approach to the question posed at the beginning of this paragraph.

Indeed, as shown in Figures 10 and 11 (that display odd products only), both in a $(2k) \times (2k)$ table and $(2k-1) \times (2k-1)$ table there are k^2 odd numbers. So, in the former case, there are $3k^2$ even numbers in a table, while in the latter case there are $3k^2 - 4k + 1$ such numbers. In other words, in an even size table the percentage of even numbers does not depend on the table's size. On the other hand, in an odd size table such percentage is described by the fraction $\frac{3k^2-4k+1}{(2k-1)^2}$, which tends to $\frac{3}{4}$ as k grows larger.

The ease of spreadsheet computing can be put to work to *mediate* the counting of other numbers with special properties in the multiplication table. So the next question could be: *How many multiples of 3 are there in the 10×10 table?* Figure 9 suggests how to find an answer to this question (without either looking at the computer-generated answer or counting the multiples of 3 one by one). Indeed, if one of the factors is a multiple of 3, a corresponding product is divisible by 3. Apparently, there are $51(= 10+10+10+10+10+10-9)$ such products in the table. Note that this result can be obtained by using tree diagram.

It should be noted that an important role that technology, in general, and a spreadsheet, in particular, as an intellectual tool can play in one's cognitive development is the emergence of the so-called residual mental power that can be used in the absence of this tool [5]. In order to assess this mental power, one has to be presented with a follow-up task when the tool is *not* immediately available for use. For example, teachers can be presented with worksheets picturing an empty 10×10 multiplication table and asked to find out as to how many numbers in this table are divisible by 8? The answer to this question can be found by pure mathematical reasoning. Indeed, if one of the factors is 8, a corresponding product is divisible by 8. Apparently, there are $19(= 10 + 10 - 1)$ such products in the table. Furthermore, when one of the factors is 4 and another factor is a multiple of 2, there exist $7(= 4+4-1)$ new products divisible by 8. This makes the total number of products divisible by 8 in the 10×10 multiplication table equal to 20. As far as one's mathematical development is concerned, there is a big difference between finding this number through spreadsheet modeling alone and arriving at this number through mathematical reasoning *mediated* by spreadsheet modeling. Now, a spreadsheet can be used to confirm this result computationally. In such a way, the use of a spreadsheet in the mathematics classroom can be the mixture of computing activities and mental tasks that are true reflections on the activities. Finally, explorations of that type may lead to the integration of the multiplication table, three diagrams, and the problems of chance.

SIZE 10 MODULO 2 TOTAL 75

	1	2	3	4	5	6	7	8	9	10
1		2		4		6		8		10
2	2	4	6	8	10	12	14	16	18	20
3		6		12		18		24		30
4	4	8	12	16	20	24	28	32	36	40
5		10		20		30		40		50
6	6	12	18	24	30	36	42	48	54	60
7		14		28		42		56		70
8	8	16	24	32	40	48	56	64	72	80
9		18		36		54		72		90
10	10	20	30	40	50	60	70	80	90	100

Figure 8: Displaying the multiples of two.

SIZE	10			MODULO	3		TOTAL	51

	1	2	3	4	5	6	7	8	9	10
1			3			6			9	
2			6			12			18	
3	3	6	9	12	15	18	21	24	27	30
4			12			24			36	
5			15			30			45	
6	6	12	18	24	30	36	42	48	54	60
7			21			42			63	
8			24			48			72	
9	9	18	27	36	45	54	63	72	81	90
10			30			60			90	

Figure 9: Displaying the multiples of three.

9 PLE 7: The sum of odd numbers in the multiplication table

Let $S_{odd}(n)$ be the sum of odd numbers in the $n \times n$ multiplication table. This sum can be found without difficulty if one observes (Figures 10 and 11) that $S_{odd}(n)$ is the same for $n = 2k-1$ and $n = 2k$. Such an observation can be confirmed computationally by constructing a two-column representation of $S_{odd}(n)$ as a function of n using a technique described in PLE 3. Furthermore, each of the k rows in a table of either size is a multiple of the first row; the sum of all numbers in the first row is $1+3+5+\ldots+(2k-1) = k^2$. In such a way, $S_{odd}(n) = k^2(1+3+5+\ldots+(2k-1)) = k^4$, where $k = INT\left(\frac{n+1}{2}\right)$.

10 PLE 8: The sum of even numbers in the multiplication table

Let $S_{even}(n)$ be the sum of even numbers in the $n \times n$ multiplication table. Whereas the sum of odd numbers in both the $(2k) \times (2k)$ table and $(2k-1) \times (2k-1)$ table equals k^4, the sum of even numbers changes as one moves from one of these tables to another. Substituting $n = 2k$ in formula (3) yields

$$S_{even}(2k) = S(2k) - S_{odd}(2k) = k^2(2k+1)^2 - k^4 = k^2(k+1)(3k+1) \qquad (7)$$

Substituting $n = 2k-1$ in formula (3) yields

$$S_{even}(2k-1) = S(2k-1) - S_{odd}(2k-1) = k^2(2k-1)^2 - k^4 = k^2(k-1)(3k-1) \quad (8)$$

HIDDEN MATHEMATICS OF THE MULTIPLICATION TABLE

SIZE	10	MODULO	2	SUM	625

	1	2	3	4	5	6	7	8	9	10	n	$S_{odd}(n)$
1	1		3		5		7		9		1	1
2											2	1
3	3		9		15		21		27		3	16
4											4	16
5	5		15		25		35		45		5	81
6											6	81
7	7		21		35		49		63		7	256
8											8	256
9	9		27		45		63		81		9	625
10											10	625

Figure 10: The sum of odd numbers in an even size table.

Formulas (7) and (8) can be confirmed computationally by using a spreadsheet (see Figures 12 and 13).

Note that having a spreadsheet that can generate numbers of different parity in the table, one can decide whether to find the sum of even numbers directly or indirectly by finding the sum of odd numbers first and then subtracting it from the sum found through formula (3). Teachers can be encouraged to develop formulas (7) and (8) first in order to see how this approach is different from the one suggested in PLE 8 in terms of the complexity of the former. The former approach, though more complex than the latter, can uncover the presence of triangular numbers in the table and demonstrate their role in the development of summation formulas.

11 PLE 10: Geometric applications and interpretations of summation formulas

Many algebraic results (summation formulas) found through exploring the above PLEs can be associated with integer-sided rectangles on a square checkerboard. For example, as was already mentioned in PLE 3, the total number of rectangles within the $n \times n$ checkerboard equals to the sum of all numbers in the $n \times n$ multiplication table [9]. By exploring the spreadsheet of Figure 8, one can see that the multiples of 2 in the $(2k)\times(2k)$ multiplication table can be associated with rectangles (on the related checkerboard) having at least one odd dimension; same numbers in the $(2k-1)\times(2k-1)$ multiplication table can be associated with rectangles having at least one even dimension. Alternatively, one can see (Figure 10) that odd numbers in the $(2k)\times(2k)$ multiplication table can be associated with rectangles having both dimensions even numbers. By the same token, in the $(2k-1)\times(2k-1)$ multiplication table odd numbers can be associated with rectangles having both dimensions odd numbers (Figure 11).

SIZE	9	MODULO	2	SUM	625

	1	2	3	4	5	6	7	8	9
1	1		3		5		7		9
2									
3	3		9		15		21		27
4									
5	5		15		25		35		45
6									
7	7		21		35		49		63
8									
9	9		27		45		63		81

n	$S_{odd}(n)$
1	1
2	1
3	16
4	16
5	81
6	81
7	256
8	256
9	625

Figure 11: The sum of odd numbers in an odd size table.

In that way, the use of technology allows one to enrich curriculum in the context of the multiplication table and to pose the following exploratory problems for teachers.

On the $(2k-1)\times(2k-1)$ *checkerboard find the total number of rectangles at least one side of which is an even number.* To this end, the results found through exploring PLE 7 can be used.

On the $2k\times 2k$ *checkerboard find the total number of rectangles at least one side of which is an even number.* To this end, a slight modification of summation techniques demonstrated in PLE 3 and PLE 7 can be used.

On the $2k\times 2k$ *checkerboard find the total number of rectangles at least one side of which is an odd number.* To this end, a combination of results found through exploring PLE 3 and PLE 7 can be used.

On the $(2k-1)\times(2k-1)$ *checkerboard find the total number of rectangles at least one side of which is an odd number.* To this end, a slight modification of summation techniques demonstrated in PLE 3 and PLE 8 can be used.

12 Concluding remarks and suggestions for further explorations

Recommendations by the Conference Board of the Mathematical Sciences for the preparation of elementary teachers include the need for courses that demonstrate how mathematics can be approached, “at least initially, . . . from an experientially based direction, rather than an abstract/deductive one” [6, p96]. Such an approach, when enhanced by the use of spreadsheets, makes it easier for the teachers to grasp the meanings of generalization and to appreciate the creation of an increasing number of general concepts on higher levels of abstraction. Indeed, as one pre-service teacher put it: “*Although frustrated at first, I was very enthusiastic to learn that strategies could build upon one*

SIZE	10	MODULO	2	SUM	2400

	1	2	3	4	5	6	7	8	9	10		n	$S_{even}(n)$
1		2		4		6		8		10		1	0
2	2	4	6	8	10	12	14	16	18	20		2	8
3		6		12		18		24		30		3	20
4	4	8	12	16	20	24	28	32	36	40		4	84
5		10		20		30		40		50		5	144
6	6	12	18	24	30	36	42	48	54	60		6	360
7		14		28		42		56		70		7	528
8	8	16	24	32	40	48	56	64	72	80		8	1040
9		18		36		54		72		90		9	1400
10	10	20	30	40	50	60	70	80	90	##		10	2400

Figure 12: The $2{\cdot}5{\times}2{\cdot}5$ multiplication table.

another to make mathematics easier. For instance, learning that the sum of all numbers on an n-multiplication chart is equal to the number of various types of rectangles on an $n \times n$ checkerboard, is something that some children may find magical (if not at least surprisingly beneficial). This assignment ... shows the complexities of math concepts, followed by the simplicity of dealing with those concepts through an equation or some other means, such as Spread Sheets." In this comment, one can see the teacher's appreciation of the logic in problem-solving strategies that are based on technology-enhanced multiple representations and readiness to bring these strategies into her own classroom.

Furthermore, by situating mathematical ideas in computationally supported environments, one can "help prospective teachers develop an understanding of the role of proof in mathematics" [6, p14], another basic recommendation for teacher preparation. In the words of another pre-service teacher: "*My understanding of the meaning of mathematical proof is that in order to prove an experiment you must have theory in order to support the experiment that is being done. This is important because, in many cases, using experiential knowledge we can solve a math problem, but without knowing exactly how we solved it.*" The teacher's remark is due to her appreciation of the role that spreadsheets can play in developing a "deep understanding" of school mathematics through "giving meaning to the mathematical objects under study" [6, p57].

Teaching ideas presented in this paper, reflecting work done with pre-service elementary teachers, can also be used in capstone courses for secondary pre-teachers. Indeed, the capstone course concept includes the need for the teachers of secondary mathematics to learn using technology as a means to reach an appropriate depth of the curriculum; in particular, to develop "understanding of ways to use ... spreadsheets as tools to explore algebraic ideas and algebraic representations of information" [6, p214] in solving complex problems. In more general terms, by focusing on numerical approach to algebra in capstone courses, one can develop the appreciation of how concrete and ab-

SIZE	9	MODULO	2	SUM	1400

	1	2	3	4	5	6	7	8	9	n	$S_{even}(n)$
1		2		4		6		8		1	0
2	2	4	6	8	10	12	14	16	18	2	8
3		6		12		18		24		3	20
4	4	8	12	16	20	24	28	32	36	4	84
5		10		20		30		40		5	144
6	6	12	18	24	30	36	42	48	54	6	360
7		14		28		42		56		7	528
8	8	16	24	32	40	48	56	64	72	8	1040
9		18		36		54		72		9	1400

Figure 13: The (2·5-1)×(2·5-1) multiplication table.

stract representations of mathematical concepts can be bridged effectively across the curriculum.

It should be noted that although possible learning environments of this paper have been organized around one of the most well known numerical structures of elementary mathematics, they are not necessarily aimed at an average student in an average pre-college classroom. The main goal of these environments, grounded in the hidden mathematics curriculum framework, is to contribute to the preparation of qualified teachers of mathematics as "the best way to raise [average] student achievement" [6, p3] at any grade level. Such qualification is also a crucial factor in realizing full potential of talented students by appreciating and nurturing their creative ideas [6]. In particular, in order to encourage and challenge such students, a teacher can use the spreadsheet-enabled journey into the multiplication table as a grade-appropriate enrichment in algebra, number theory, and discrete mathematics.

To conclude, note the ideas of this paper can be extended to other two-dimensional arrays of numbers generated by partial difference equations in two variables. The addition table is the simplest such example. More complicated arrays to explore using the hidden mathematics curriculum framework enhanced by the power of spreadsheet modeling may include addition/multiplication tables in different base systems (or modular arithmetic), polygonal numbers, and binomial coefficients, to name just a few. A challenge would be to find the application of algebraic results associated with these arrays to counting problems like the one presented in [9] and those of the previous section.

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