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Colour by Numbers: Solving Algebraic Equations Without Algebra

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Colour by Numbers: Solving Algebraic Equations Without Algebra

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Abstract

A series of Excel models is described. They allow solution of *f(x) = 0* by examining change of sign of function values. The function is graphed and change of sign easily detected by a change of colour. Prominent Excel features used are *Names*, *Chart Wizard* and *Conditional Formatting*. Several sample Excel models are available for download, and the paper is intended to be used as a lesson plan for students having some familiarity with derivatives. For comparison and reference purposes, the paper also presents a brief outline of several common equation-solving strategies as an Appendix.

Keywords: solving equations, Excel, conditional formatting.

1 Introduction

Many algebraic equations cannot be solved by algebra; for example, $2^x = x^2$. One way to find an approximate solution to algebraic equations of the form *g(x) = h(x)* is to accurately graph *y = g(x)* and *y = h(x)* as separate functions of *x*. Any points of intersection will then be solutions. A better method, now that we have computers to do all the tedious calculations, is to first express the equation in the standard form *f(x) = 0*, and then to determine points where the graph intersects the *x*-axis. This standard form is usually obtained by transposing the function on the right to the left side of the equation, that is, *g(x) – h(x) = 0.* Then we take *f(x) = g(x) – h(x)*. For our example above, we would have $f(x) = 2^x - x^2$.

We now consider the use of function plotting in Excel to locate solutions of *f(x) = 0* to essentially any desired accuracy using a graphical *zoom-in* process. Automatic change of colour of ordinates in Excel is used to drive the process.

1.1 Why construct a graph? How many solutions are there?

Some equations have just one solution: for example, *3x – 12 = 0*. Other equations have many solutions. For example, the equation sin *x = 0* has an infinite number of solutions. In other cases, there may be no solutions at all. A simple example of an equation with no real number as solution is $x^2 + 1 = 0$. Our example just above, $2^x = x^2$, has three solutions. Before attempting any computer-based solution method, it is always useful to know if our equation has a solution or not! Usually, the best way see do this is to construct a graph of *y = f(x)*. In the case of $x^2 + 1 = 0$, the graph will be entirely above the *x* axis, so we know that there is no solution. In many instances, the graph will be good enough to determine the

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solutions, provided we have some reasonably efficient way of isolating, or *zooming-in* on each solution. This is our aim in this paper.

1.2 What about algebra?

We have said that algebra is not adequate to solve some equations. Even if we can solve a given equation by algebra or other method, it is always a good idea to have some reasonably-accurate plot of the function before we start. This gives us some idea of the general shape of the curve, and may also point out some region where problems may arise for our particular method of solution. The graph of the function $f(x) = (x - 2)(x - 5)$ crosses the axis at $x = 2$ and $x = 5$. In this case, we can tell by inspection of the form of the function that the solutions are 2 and 5. However, this is a rare luxury. If we draw the graph of y = f(x), then from a geometrical point of view, we are looking for points where the function intersects the x-axis.

1.3 Can we get *f(x)* to be exactly zero?

We wish to find a value of x such that $f(x) = 0$. For example if $f(x) = x^3 - 10x + 2$, then we are solving the equation $x^3 - 10x + 2 = 0$ for x. It is important to note that, in general, we are not expecting an exact answer for x, but one which makes the function very small in magnitude (small, but could be either negative or positive). For example, $x = 3.05709$ gives $f(x) \approx 5 \times 10^{-5}$, which may be small enough to accept this value of x as a reasonable approximation to a solution. In the graphical method we consider here, approximate solutions are what we seek. Occasionally we might get an exact solution (we shall see that this occurs in our example, $2^x = x^2$, where two solutions are integers, and one is not).

1.4 Simple function plotting in Excel

We want Excel to plot the graph of $y = f(x)$ for us on a suitable interval $a \le x \le b$ with a minimum of fuss. Fortunately, with a few basic principles of spreadsheeting and Excel's Chart Wizard, it is not too hard at all.

First, we need a table of ordered pairs (x, y). These represent points on the graph of $y = f(x)$. Excel is just perfect for generating these, and then it's pretty reasonable at plotting them too. In the language of multimedia and digital audio, we are *sampling the function* (putting various values of x into the function formula to compute corresponding y values, and then tabulating them). Then, for plotting these points, Chart Wizard is the way to go. First, we consider a simple example, and then consider a more flexible approach, which will allow us to easily change the plot interval for x.

2 Function plotting example

Let's now graph the function $y = x^2$ on the interval $-3 \le x \le 3$ in Excel. We use just 30 points (actually 31 points if we want to have -3.0, -4.8, …, 4.8, 3.0; check this carefully!). We now follow these steps.

1. Start with a new Excel worksheet.
2. First put the headings **x** and **y** in cells A1 and B1 respectively; make these two cells centred and bold.
3. Now select the range A1:B32, and then **Insert:Name:Create**. Excel will suggest the top row, which is slightly misleading, but it does what we need. It is actually creating the names for your selection in *everything but the top row*. It will create names a and b, *based on the top row*, but the names refer to the respective ranges in columns A and B, minus the heading (top row). If you want to see what has happened, click on a name in the Name box (just above cell A1), or select the menu option **Insert:Name:Define**.
4. Now put the value -3 in A2, the value -2.8 in A3, then select the range A2:A3, grab the fill-handle and fill down to A32.
5. In cell B2, type =x*x. Double-click on the fill-handle of B2 and watch what happens. Excel will fill up the column of ordinates (column B).
6. We now have our table of *(x, y)* pairs, so we can get Excel to plot our graph. First, it is advisable to select our tabulated pairs. There are easy and hard ways to do even this! One of the simplest is to make use of the name box. Go to the name box, select *x*; then hold down the CTRL key while going back to the name box and selecting y.
7. Now click on the chart wizard button. We want a scatter diagram, so select this one, with the "Scatter with data points connected by smoothed lines without markers" sub-option. Always choose the *scatter* chart type for "*y = f(x)*" style mathematical function plots.
8. Click Finish.
9. At this point you should have a standard parabolic graph. Select the text box which say "Series1" and delete it.
10. We make some cosmetic improvement to the graph; such as making it slightly thicker. Right-click on the curve somewhere and select **Format Data Series**, then select the **Patterns** tab. Adjust the weight parameter to next thicker one.
11. Our background is a drab grey, so right-click on the grey background and select **Format Plot Area**. Choose a pleasing pastel shade such as very light yellow. Your graph should look like Figure 1. We have made ours a little narrower, so yours may look wider. That's OK!
12. Save your model as 01 Simple plotting.xls.

Plotting this function in Excel was no problem, except that if we change our minds on the plot interval, we must do most or all of those steps again! In the next section, we consider an improved method.

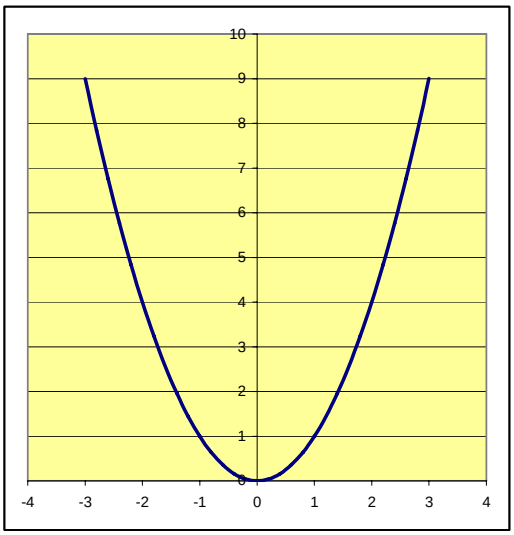

Figure 1: Simple Plotting

3 An improved approach to plotting *y = f(x)*

Our model is quite primitive, because if we wished to observe our function plotted over a different interval, we need to mess around with the *x* values in column A. It is preferable to set up some plotting parameters, as shown in Table 1. They are intended to allow plotting of a function described by a single formula *y = f(x)* over the interval $a \le x \le b$, using *n* subdivisions, of equal width, *h*. The parameter *h* is the step size, i.e., the horizontal distance between neighbouring values of *x*. This process generates an arithmetic sequence of values of *x*:

$$x_0 = a + 0h$$
$$x_1 = a + 1h$$
$$x_2 = a + 2h$$
$$\ldots$$
$$x_n = a + nh = b$$

Here are the steps:

13. Type the headings a, b, n, h in cells C1 to F1 respectively. Make them centred and bold like the x and y headings.
14. Select the range C1:F2 and choose the menu option **Insert:Name:Create**. As in step 3, Excel will create names for your parameters, using your headings as a guide.
15. In C2 type the value -3; in D2 the value 3; in E2, the value 30, and in F2 the formula "=(b – a)/n".
16. Now replace the first x value in A2 by the formula "=a".

17. In A3, type "=", click in A2, then type "+ h". Double-click on the fill-handle for A3 and the x values in column A will be updated. Now, to zoom in or out on a plot, we just change the value(s) of *a* and/or *b*.
18. We now look at a different function. In B2 type the formula "=2^x – x*x", then double-click on B2's fill-handle; Excel will fill column B with the new formula.
19. Now change the plot interval to $a = -2$ and $b = 5$ by typing -2 in cell C2 and 5 in cell D2. Notice that the x and y values are automatically recomputed and the graph re-plotted. We see that this function crosses the x axis three times on the interval $-2 \le x \le 5$.
20. Save your model as *02 Better plotting.xls*. It should look something like Figure 2.

Table 1: Plotting parameters

Symbol	Meaning	Value/formula
a	Left endpoint	-3
b	Right endpoint	3
n	Number of sub-intervals	30
h	Step size	$(b-a)/n = 0.1$

	A	B	C	D	E	F	G	H	I
1	**x**	**y**	**a**	**b**	**n**	**h**			
2	-2	-3.75	-2	5	30	0.233333			
3	-1.76667	-2.82722							
4	-1.53333	-2.00563							
5	-1.3	-1.28387							
6	-1.06667	-0.66036							
7	-0.83333	-0.13321							
8	-0.6	0.299754							
9	-0.36667	0.641128							
10	-0.13333	0.893945							
11	0.1	1.061773							
12	0.333333	1.14881							
13	0.566667	1.159986							
14	0.8	1.101101							
15	1.033333	0.97897							
16	1.266667	0.801606							
17	1.5	0.578427							
18	1.733333	0.320507							
19	1.966667	0.040862							
20	2.2	-0.24521							
21	2.433333	-0.51971							
22	2.666667	-0.76151							
23	2.9	-0.94574							
24	3.133333	-1.04318							
25	3.366667	-1.01948							
26	3.6	-0.83427							
27	3.833333	-0.44006							
28	4.066667	0.218928							
29	4.3	1.208311							
30	4.533333	2.605197							
31	4.766667	4.500238							
32	5	7							

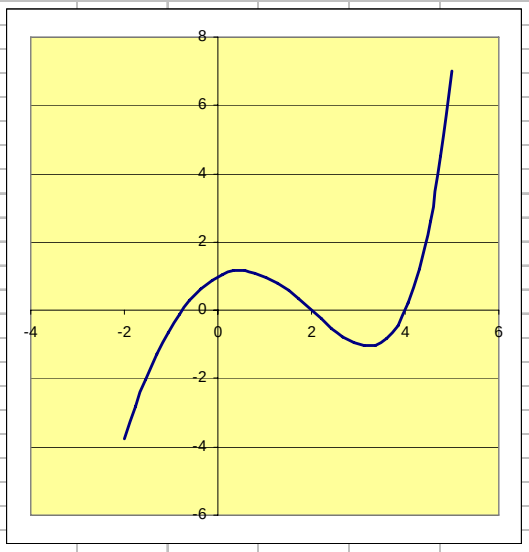

Figure 2 : Better Plotting

For this plotting model in Excel, we have a reasonable degree of flexibility. One thing we cannot easily change is the number of sample points of the function; this is 31 here, but this is often adequate.

4 Colour by numbers

We now give our model a very useful enhancement, with only a few more steps. Excel has a feature called *conditional formatting*, whereby certain aspects of a cell's format (font, colour or border) may be made to automatically depend on the current value stored in the cell. We can make good use of this feature by asking negative, zero and positive values to all have different colours. Fortunately, Excel allows up to three formatting conditions – just the number we need. Here are the steps.

21. Go to the Name Box and select *y*.
22. Select the menu option **Format:Conditional Formatting**. A dialog box will appear.
23. Here, we use some pastel shades to automatically indicate the sign of *y*: light brown or light orange for negatives, green for zero, and blue for positives (simple rationale: underground, grass, and sky). In the drop-down list-box to the right of "Cell Value Is", click on the arrow and select *less than*, tab to the next box and type *0* (zero). Then click on the *format*

button, click on the *patterns* tab, then on the lightest colour in the "orange" column (usually the second column). Click OK.

24. You have returned to the main **Conditional Formatting** dialog. Click on the *Add* >> button. The dialog box expands, allowing you to enter a second condition. Make this condition *equal to zero*, and the colour a bright green (second from the bottom in the fourth column). Click OK.

25. Repeat the previous step to get a third condition, *greater than zero*, which you should colour a light blue.

26. Close the **Conditional Formatting** dialog by clicking OK. You will notice only blue and orange/brown y values for the current function.

27. Now change the values of a and b to -3 and 3 respectively. Since the abscissa $x = 2$ is now part of the list of values in column A, we have a highlighted cell (A27) where the function is exactly zero.

28. Now change the values of a and b to 2 and 5 respectively. We now see two integer solutions: $x = 2$ and $x = 4$. It is clear that we can now look at the function over pretty much any interval we want just by changing the values of a and b.

29. Save your worksheet as *03 Colour by Numbers.xls*. It should look like Figure 3.

	A	B	C	D	E	F	G	H	I
1	**x**	**y**	**a**	**b**	**n**	**h**			
2	-2	-3.75	-2	5	30	0.233333			
3	-1.76667	-2.82722							
4	-1.53333	-2.00563							
5	-1.3	-1.28387							
6	-1.06667	-0.66036							
7	-0.83333	-0.13321							
8	-0.6	0.299754							
9	-0.36667	0.641128							
10	-0.13333	0.893945							
11	0.1	1.061773							
12	0.333333	1.14881							
13	0.566667	1.159986							
14	0.8	1.101101							
15	1.033333	0.97897							
16	1.266667	0.801606							
17	1.5	0.578427							
18	1.733333	0.320507							
19	1.966667	0.040862							
20	2.2	-0.24521							
21	2.433333	-0.51971							
22	2.666667	-0.76151							
23	2.9	-0.94574							
24	3.133333	-1.04318							
25	3.366667	-1.01948							
26	3.6	-0.83427							
27	3.833333	-0.44006							
28	4.066667	0.218928							
29	4.3	1.208311							
30	4.533333	2.605197							
31	4.766667	4.500238							
32	5	7							
33									

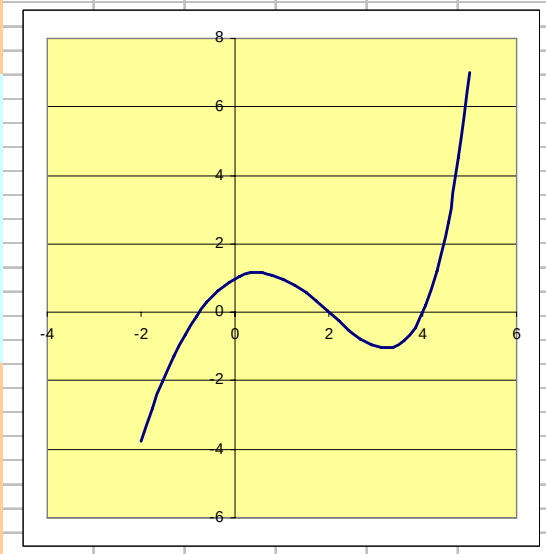

Figure 3: Colour by Numbers (CBN)

4.1 Beware

A word of caution about computer graphs: beware. Any computer-generated graph is derived from a finite number of samples of the function, e.g., in ours the function is evaluated 31 times, and these *y* values (ordinates) are plotted against the corresponding values of *x*. Since we are dealing with real-valued functions of a real variable, we have an infinite collection of function values, even in the smallest interval. In some cases, some interesting behaviour is masked by only taking a sample of these points. If you suspect something is going on, but the graph does not reveal it, then one strategy is to zoom in and get a closer look. Sometimes we need to zoom out, and get the big picture. In our Excel model, zooming is controlled by adjusting the values of *a* and *b*, the end-points of our plot interval.

5 Using our model to solve *f(x) = 0*

By noticing a change of colour on the tabulated ordinates, we can quickly spot a region where the function is zero. The only way for a function to change sign without being zero in between is for it jump across the *x* axis. This will be the case if the graph of the relationship y = *f(x)* is *discontinuous*. Such functions certainly exist, but we do not consider them here. Therefore, if we spot a change of colour, we can be certain that our function has passed through zero.

Sometimes functions have "repeated roots". This means that, at a certain value of *x*, not only is the function zero, but so is the first derivative (and perhaps also higher derivatives). Simple examples of this occur for $y = x^n$ when $n = 2, 3, 4 \ldots$. We normally look for change of sign of the function to find roots, but such repeated roots can complicate matters. It will usually be obvious from the graph just what situation we are dealing with.

Let's now use our model to find the third solution of $f(x) = 2^x - x^2 = 0$.

30. Click on B2 and alter the formula for the function to be = 2^x – x^2.. Double click the fill handle for B2 to fill the new formula into all the y-values
31. Change the value of a (C2) to be -1 and the value of b (D2) to be 5.

This "bigger picture" shows us that there is a solution between -1 and 0. Your graph should now look like Figure 4.

	A	B	C	D	E	F	G	H	I
1	**x**	**y**	**a**	**b**	**n**	**h**			
2	-1	-0.5	-1	5	30	0.2			
3	-0.8	-0.06565							
4	-0.6	0.299754							
5	-0.4	0.597858							
6	-0.2	0.830551							
7	0	1							
8	0.2	1.108698							
9	0.4	1.159508							
10	0.6	1.155717							
11	0.8	1.101101							
12	1	1							
13	1.2	0.857397							
14	1.4	0.679016							
15	1.6	0.471433							
16	1.8	0.242202							
17	2	0							
18	2.2	-0.24521							
19	2.4	-0.48197							
20	2.6	-0.69713							
21	2.8	-0.8756							
22	3	-1							
23	3.2	-1.05041							
24	3.4	-1.00394							
25	3.6	-0.83427							
26	3.8	-0.51119							
27	4	0							
28	4.2	0.739174							
29	4.4	1.752127							
30	4.6	3.091465							
31	4.8	4.817618							
32	5	7							
33									
34									

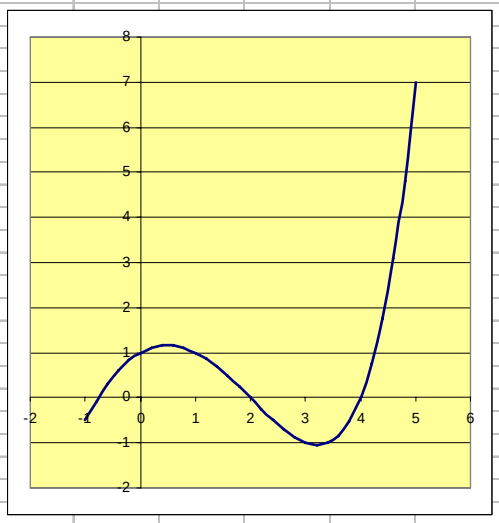

Figure 4: CBN showing two solutions of $f(x) = 2^x - x^2 = 0$

We still have our two integer solutions, viz., $x = 2$ and $x = 4$. This may be seen both from the graph and also from the tabulated y values when $x = 2$ and $x = 4$, since they are both green.

Between the cells B3 and B4, you should see a change of colour in the y column between x = -0.8 and x = -0.6. This means that there is a solution between these x values. Here is an important step: if we now change the values of a and b to -0.8 and -0.6 respectively, this will effect a zoom-in operation, on the region containing the solution. Do this and see what happens! Your new colour change will be between $x = -0.76667$ and $x = -0.76333$.

Note that we have this solution, by changing the values of a and b just once, in order to effect a *zoom-in* operation. This operation can be repeated as often as desired to get much greater accuracy.

Note also that no algebra has been required; merely typing the formula for $f(x)$ in cell B2, and double-click for fill-down. The rest is observation of graph and colours. To do the zoom-in, we have to type in new values of a and b, but no algebra is required.

5.1 Comments

The technique just described works for essentially any continuous function, and not just quadratics or polynomials. We call it ***Colour by Numbers***. Except for pathological cases, most common functions look linear on a small enough interval. This may be seen by zooming in. The techniques described may be extended to

determine maximum and minimum values of functions, by examining derivative(s) of the function of interest. We consider this in the next section.

6 Derivatives

Adding estimates of first and second derivatives to our model is quite simple. This helps us to locate local maxima and minima. We use estimates with accuracy $O\left(h^2\right)$. They are:

$$f'(x) \simeq \frac{f(x+h) - f(x-h)}{2h}$$

and

$$f''(x) \simeq \frac{f(x+h) - 2f(x) + f(x-h)}{h^2}$$

In these formulas h is just the same h we have been using to separate our x values. Here are the steps.

32. Select column B. Right-click and select **Insert**. Excel makes room for a new column. Now hit F4 to get another new column.
33. Select both columns B and C and then make the menu selection **Data:Group** and **Outline:Group**. We can now hide the derivatives later if they are of no interest.
34. In B1 type "dy/dx" and in C1 type "d^2y/dx^2". The superscripted 2 can be achieved by selecting the 2 and selecting Format: Cells: Font and checking the Superscript box. In B2, C2, B32, C32 type the word "estimate". This reminds us that our derivatives are not exact. They will be quite accurate, however, as we have used second-order estimates. Also, it will stop us from adding the derivative formulae to cells in which it cannot be calculated, as both the formulae use values in the y-column that are above and below the current cell.
35. In cell B3, type the formula "=(D4 - D2)/2/h" and double-click to fill down.
36. In cell C3, type the formula "=(D4-2*D3+D2)/h/h" and double-click to fill down.
37. Now select a cell with any y value in column D. Click on the Format Painter button, and paint the range B3:C31 with this format. You will get the conditional formatting colours now for derivatives as well as ordinates.
38. Save your worksheet as *04 Colour by Numbers with Derivatives.xls*.

7 Automation of the model

While the model just presented does a reasonable job, we would like to get Excel to automatically update *a* and *b* instead of the user having to key them. An extended model, 05 Colour by Numbers with Derivatives & AutoZoom.xls, is available for download. The user simply selects a range of cells in column A, then clicks on the *Zoom In* button. A *Zoom Out* button is also provided, which simply doubles the length of the plot interval, while maintaining its midpoint.

This model has some VBA code to implement the Zoom functionality behind the two buttons. For this to run properly, you need macros to be enabled, and you will be prompted to give permission for this when the model loads. You also need macro security level set to Medium. This may be altered as required by selecting the menu option **Tools:Options**, then the tab **Security**, then clicking on the button **Macro Security**. You need to have this set to Medium or lower for the VBA code to run.

x	dy / dx	d^2y / dx^2	y
-1	estimate	estimate	-0.5
-0.8	1.999385	-1.72361	-0.0657
-0.6	1.658773	-1.682511	0.29975
-0.4	1.326992	-1.635301	0.59786
-0.2	1.005354	-1.581071	0.83055
0	0.695369	-1.518777	1
0.2	0.39877	-1.44722	1.1087
0.4	0.117546	-1.365023	1.15951
0.6	-0.14602	-1.270602	1.15572
0.8	-0.38929	-1.162142	1.1011
1	-0.60926	-1.037554	1
1.2	-0.80246	-0.89444	0.8574
1.4	-0.96491	-0.730045	0.67902
1.6	-1.09203	-0.541205	0.47143
1.8	-1.17858	-0.324284	0.2422
2	-1.21852	-0.075108	0
2.2	-1.20492	0.2111201	-0.2452
2.4	-1.12982	0.53991	-0.482
2.6	-0.98407	0.9175904	-0.6971
2.8	-0.75717	1.3514313	-0.8756
3	-0.43704	1.8497837	-1
3.2	-0.00984	2.4222402	-1.0504
3.4	0.540364	3.07982	-1.0039
3.6	1.231864	3.8351809	-0.8343
3.8	2.085669	4.7028627	-0.5112
4	3.125912	5.6995673	0
4.2	4.380316	6.8444803	0.73917
4.4	5.880728	8.15964	1.75213
4.6	7.663729	9.6703617	3.09147
4.8	9.771337	11.405725	4.81762
5	estimate	estimate	7

a	b	n	h
-1	5	30	0.2

Zoom In Zoom Out

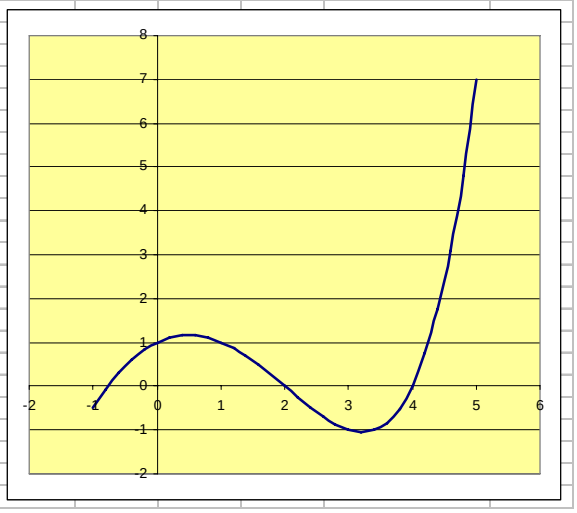

Figure 5: Colour by Numbers with Derivatives & AutoZoom.xls

8 Closing remarks

Many other numerical methods exist to solve $f(x) = 0$, but whichever one is chosen, it is usually best to graph the function so that an overall picture of its behaviour can be seen. As we have noted, this can be done in a very straightforward manner in Excel.

Numerical methods are typically iterative (they converge on a solution after series of steps) and require an estimate of x to get started. This can usually be obtained by inspection of the graph of the function, giving us yet another reason for having this at our disposal.

As we noted earlier, graphing functions was extremely tedious and error-prone before the advent of computers. With tools such as Excel so readily available, there is really no longer any excuse for failing to generate the graph of $y = f(x)$ before attempting to solve $f(x) = 0$.

9 Exercises

Use the supplied models to find all zeroes, local maxima and minima and inflection points of the following functions over the given intervals. Find each point correct to 4 decimal digits by zooming in.

1. $y = 1 - x + 2x^2$ on $-5 \le x \le 5$
2. $y = 1 + x\sqrt{8 - x} - x^2$ on $0 \le x \le 5$
3. $y = e^x x^2 - 5$ on $-4 \le x \le 2$
4. $y = x^3 - x^2 + x - 2$ on $-5 \le x \le 5$
5. $y = 1 - 3/(x^2 + 2)$ on $0 \le x \le 2$
6. $y = \sin(1/x)$ on $-1 \le x \le 1$
7. $y = x^3 - 3x + 2$ on $-3 \le x \le 3$
8. Solve $|4x + 5| = x^2 - 6x + 2$ for x. Use an algebraic method and an Excel method.
9. Solve $x^2 = 2^x$ for x in Excel. Find all three solutions correct to six decimal places each.
10. Use an Excel model to determine k so that the equation $\sqrt{x} = kx + 20$ has a solution in $0 \le x \le 4$, and find that solution correct to 4 decimal digits.

Appendix: Some strategies to solve equations

We outline some ways of approaching the problem of solving *f(x) = 0* for *x*.

Strategy 1: Guess a solution. This is also called *trial and error*. It is a technique where you simply substitute selected values of *x* and see if you get zero. Of course, the question of which values of *x* to select is left unanswered. This approach used to be very tedious and error-prone, but now we have spreadsheets like Excel and it is very easy. Though trial and error sounds very unscientific, it underlies many computer methods, which conduct a (more systematic) search for *x*. Example: $f(x) = x^3 - 10x + 2$. We try $x = 3$ and get $f(3) = -1$. We try $x = 4$ and get $f(4) = 26$. Neither of these is zero, so we haven't found a solution, but since the function changes sign as *x* goes from 3 to 4, then there must be a solution in between (probably a lot closer to 3 than to 4 – why?). We could build up a table of values of *x* with corresponding function values. This is useful as we can observe "trends" and it leads to some of the other methods, as described below.

Strategy 2: Algebra. This method will work for very simple equations such as *5x – 23 = 52*, in this case leading to the solution *x = 15*. Although algebra is extremely useful, we rapidly run into brick walls when we try to apply it to solve equations that occur in many practical problems. Most models lead to equations which are not linear, and, while it is possible to solve some more complicated (but very specialized) equations by algebra, the vast majority of real-world problems do not yield to this approach. For example, we can solve a *quadratic equation* by the algebraic process of completing the square, or, what is really the same thing, by use of the quadratic formula (see below). Beyond this, very little progress on solving equations can be made until so-called *numerical methods*, i.e., computer-based methods, are introduced. Some of these are very briefly described below.

Strategy 3: Use a special formula. The range of equations that may be solved with this technique is extremely small. The most well-known is the quadratic. If we wish to solve $x^2 - 7x + 12 = 0$, then the so-called quadratic formula gives the solution in terms of the coefficients:

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

In our case, application of this formula yields $x = 3$ or $x = 4$.

Strategy 4: Factorization. This is a variation of the algebraic approach and relies on the fact that if two quantities are multiplied together and the result is zero, then at least one of them must be zero. Thus, to solve $(x-3)(x+7) = 0$ we can argue that either $(x-3)$ or $(x+7)$ must be zero. This means that $x = 3$ or $x = -7$. This is a very fundamental and powerful technique, but it assumes that the function we are dealing with can be factorized. Most functions cannot. For many problems, effort spent in attempted factorization is better spent on other approaches.

Strategy 5: Construct a graph of *y = f(x)* and observe where *y = 0*. This method extends the first strategy of trial and error. We select an interval (range of x values) over which we wish to evaluate and plot (sample) the function. It is simple but effective, and easily implemented in Excel.

Strategy 6: Bisection method. This method also extends the first strategy of trial and error. We construct the graph as in the previous method, and try to locate an interval over which the function changes sign. We cut this interval into two equal halves (bisect it) and then choose the half on which we still have a change of sign of the function. This process is repeated until the interval is so narrow that any point in it will serve as our answer. Again, the method is easily implemented in Excel.

Strategy 7: Newton's method. This is a very powerful technique, usually giving very high accuracy with only a few steps, but it requires use of derivatives. It is also very easily implemented in Excel, provided we have some way of evaluating, or at least estimating, the derivative (rate of change of the function). A sequence of iterates may be generated by the formula:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

We need to somehow find a starting value x_0, and this is best done by inspecting the graph of *y = f(x)* in Excel. Newton's method may fail, but when it converges, it usually does so with great speed.

Strategy 8: Secant method. This is a modification of Newton's method so that we do not have to compute derivatives. It is also easily implemented in Excel. Unlike Newton's method, the secant method requires two values to get started. We call them x_0 and x_1. The formula for the secant method is:

$$x_{n+1} = x_n - \frac{f(x_n)\left(x_n - x_{n-1}\right)}{f(x_n) - f(x_{n-1})}$$