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# Discrete Phase-Locked Loop Systems and Spreadsheets

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This paper demonstrates the use of a spreadsheet in exploring non-linear difference equations that describe digital control systems used in radio engineering, communication and computer architecture. These systems, being the focus of intensive studies of mathematicians and engineers over the last 40 years, may exhibit extremely complicated behavior interpreted in contemporary terms as transition from global asymptotic stability to chaos through period-doubling bifurcations. The authors argue that embedding advanced mathematical ideas in the technological tool enables one to introduce fundamentals of discrete control systems in tertiary curricula without learners having to deal with complex machinery that rigorous mathematical methods of investigation require. In particular, in the appropriately designed spreadsheet environment, one can effectively visualize a qualitative difference in the behavior of systems with different types of non-linear characteristic.

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# 1 Introduction

The use of an electronic spreadsheet to support the exploration of topics in exact sciences and engineering through computing is well documented. A monograph edited by Filby [16] includes well-chosen examples from a number of areas of contemporary science and engineering, including biology, chemistry, geology and electronics. Kreith and Chakerian [21] focus on the use of a spreadsheet in modeling iterative mathematical structures ranging from Fibonacci numbers to fractals. Neuwirth and Arganbright [36], using a bicycle metaphor as a reference to software simple in structure yet powerful in educational applications, presented a spreadsheet as a user-friendly and self-sufficient exploratory tool enabling mathematical modeling of various concepts studied across the secondary and tertiary mathematics curriculum. Silva [43] used a spreadsheet as a modeling and simulation tool for studying electrical circuits. El-Hajj with co-authors [13], [14] used a spreadsheet as a tool for the simulation and modeling of various systems of automatic control, both linear and nonlinear, as a way of avoiding the use of complicated mathematical methods in the context of engineering education. Levin and Talis [30] used a spreadsheet in the context of teaching academically challenging topics in digital design associated with concurrent error detection.

The material of this paper stems from a number of educational applications of the software at the tertiary level, including its use in discrete mathematics course at Bond University, in secondary mathematics teacher education at SUNY Potsdam, and in the framework of collaboration between Intel Corporation and the Faculty of Mathematics and Mechanics of Saint Petersburg State University on a project "Control in the Distributed Systems of Clock Pulse Generators in Multiprocessor Clusters." One of the topics included in the project is the research and development of mathematical theory of discrete phase locked loops for array processors commonly used in radio engineering, communication, and computer architecture [24]. Such digital control systems exhibit high efficiency in eliminating *clock skew* (see below) – an undesirable phenomenon arising in parallel computing. To clarify, consider Figure 1 in which clock $C$ sends pulses via conduit $L$ to parallel processors $P_k$. The work of parallel algorithms requires that processors involved perform certain operations simultaneously. However, each clock pulse travels a different distance in time to drive each processor. As a result, a time-discordance in the activation of the processors arises. This phenomenon is called *clock skew*. Digital phase-locked loops have gained widespread recognition and preference over their analog counterparts because of their ability to deal with this phenomenon effectively. From a mathematical perspective, this gives rise to a problem associated with the analysis of global stability of non-linear difference equations that serve as mathematical models of the phase-locked loops [25]; that is, the analysis can be formulated in terms of parameters for such systems.

It should be noted that the study of difference equations (alternatively, iterative maps) has a long and fascinating history. Leonardo Pisano (a 13th century Italian mathematician better known by his nickname Fibonacci) was probably the first to study

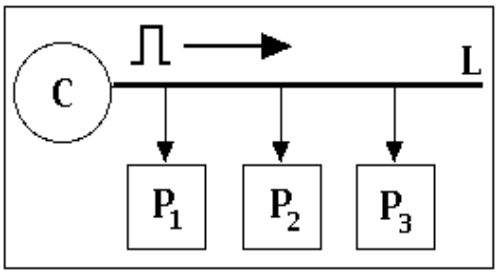



Figure 1: Travel of clock pulses.

the relation (a second order difference equation)

$$x(t+2) = x(t+1) + x(t) \qquad t \in N \tag{1}$$

satisfying the initial conditions $x(0) = x(1) = 1$ (hereafter, $N$ denotes the set of natural numbers $\{0, 1, 2, 3, \ldots\}$) in connection with his investigations of the growth of the population of rabbits breeding in ideal circumstances. In 1843, a French mathematician Jacques Binet showed that the solution to equation (1) satisfying the given initial conditions is given by the closed representation of eq (2), for $t \in N$.

$$x(t) = \frac{1}{\sqrt{5}} \left( \left( \frac{1+\sqrt{5}}{2} \right)^{t+1} - \left( \frac{1-\sqrt{5}}{2} \right)^{t+1} \right) \tag{2}$$

This result was known, however, more than a century earlier to Euler and Daniel Bernoulli [44] as well as to de Moivre [20]. Despite its rather complex form (resembling a solution to a linear differential equation of the second order), the sequence $x(t)$ defined by (2) is comprised of integers only, nowadays commonly known as Fibonacci numbers.

In 1845, a Belgian mathematician Pierre Verhulst introduced the following non-linear difference equation of the first order

$$x(t+1) = rx(t)(1 - x(t)) \qquad t \in N \qquad r > 0 \tag{3}$$

as a mathematical model of population dynamics within a closed environment that takes into account internal competition [38]. Equation (3) is commonly referred to as the *logistic equation* [44]. Since then, equation (3), which can be generalized to the form

$$x(t+1) = f(x(t)) \qquad x \in R \qquad t \in N \tag{4}$$

has been studied intensively. However, only in the second part of the $20^{th}$ century, due to the efforts of many mathematicians, the limiting structure of solutions to equation (4) has come to be recognized as extremely complex [41], [31], [42], [34], [35], [15]. In particular,

in equation (3), period-doubling bifurcations were discovered. Surprisingly, whereas solutions to a linear multidimensional discrete equation $x(t+1) = Ax(t)$, $x \in R^n$, $t \in N$ and its continuous analogue $dx/dt = Ax(t), x \in R^n, t \in R$ (here, $A$ is a constant $n \times n$ matrix), to a large extent, as formula (2) may suggest, possess similar behavior, solutions to equation (4) and its continuous one-dimensional analogue $dx/dt = f(x(t)), x, t \in R$, bear qualitatively different structure.

Another example of a non-linear difference equation important in applications is the following equation of the first order

$$x(t+1) = x(t) - \alpha \sin x(t) + \gamma \qquad t \in N \tag{5}$$

where $\alpha$ and $\gamma$ are non-negative parameters. Over the last 40 years, many authors conducted rigorous studies of equation (5) both as a pure mathematical object [6], [19] and as a mathematical model of a phase-locked loop [37], [17], [33], [32]. Osborne's [37] pioneering use of exact methods, such as the Contraction Mapping Theorem, applicable to the direct study of non-linear effects in digital phase-locked loops described by equation (5) with $\gamma = 0$, enabled for the discovery of inadequacy of the methods of linearization as means for understanding their complex behavior. However, even the exact methods used by Osborne, while revealing what then appeared as multiple cycle slipping followed by a divergent behavior of iterations, did not allow for a precise interpretation of non-linear effects discovered as transition from global asymptotic stability to chaos through period-doubling bifurcations [27]. In particular, in the latter paper, the values of parameter $\alpha$ in equation (5) with $\gamma = 0$ that correspond to period-doubling bifurcations have been found by using a combination of analytical and computational methods. In the present paper, a spreadsheet is used as a self-sufficient mathematical/pedagogical tool allowing for both finding these values and further refining them.

## 2 Global asymptotic stability as a desirable regime in phase-locked loops

Consider the difference equation

$$x(t+1) = x(t) - \alpha \sin(x(t)) \qquad t \in N \quad \alpha > 0 \tag{6}$$

which serves as a mathematical model of the simplest discrete phase-locked loop without filters with a sinusoidal characteristic of its phase detector and zero frequency misalignment. Equation (6), being a recurrence relation, has a unique solution for any initial value $x(0)$. The purpose of control in digital phase-locked loops is the elimination of a clock skew for almost all initial values $x(0)$ so that $x(t)$ tends to a stationary solution of equation (6). By equating $x(t+1) = x(t)$, one can conclude that equation (6) have two stationary solutions, $x(t) = 2j\pi$ and $x(t) = (2j+1)\pi$. As demonstrated in [27], the former solution is asymptotically stable and the latter solution is Lyapunov unstable. Therefore, whereas the unstable solution can not be realized physically, the elimination of clock skew in the phase-locked loop described by equation (6) can be guaranteed if

the relations $\lim_{t\to+\infty} x(t) = 2j\pi$ and $\lim_{t\to+\infty}(x(t+1) - x(t)) = 0$ hold true for almost all orbits $x(0)$. In turn, such property of solutions ensures a global asymptotic stability of equation (6).

As mentioned in [26], the global asymptotic stability of equation (6), in mathematical terms is equivalent to the existence of a function defined on its solutions and satisfying several conditions. One such condition requires the function's derivative to be negative. It can be shown [27] that the fulfillment of this condition implies the inequality $\alpha < 2$. Thus equation (6) is globally asymptotically stable for any $\alpha \in (0, 2)$.

The global asymptotic stability of equation (6) as a physical phenomenon can be demonstrated in a variety of forms within a spreadsheet by modeling equation (6) as a recurrence relation both numerically and graphically. First, this modeling may include the construction of a bifurcation diagram which shows the limiting behavior of iterations as parameter $\alpha$ varies along the interval $(0, 2)$ and then extends beyond it. Second, based on the results of modeling through the construction of this diagram, one can narrow the region of parameter $\alpha$ and explore equation (6) as a recurrence relation. In what follows, a spreadsheet will be presented as a user-friendly tool in modeling equation (6) for different values of parameter $\alpha$. In such a way, as several authors have argued regarding the effect of technology on curriculum [18], [9], [3], embedding advanced mathematical ideas in a technology tool like a spreadsheet enables for the investigation of problems without learners having to deal with complex machinery of non-linear control theory that mathematically rigorous methods of investigation require.

## 3 Spreadsheet-based approach to mapping a segment into itself

The problem of mapping a segment into itself that will be discussed in this section in connection with the behavior of orbits defined by equation (6) is a special case of a more general problem of mapping the circle into itself. In the specific case of phase-locked loops, this more general problem was studied in [39], [40], [7]. Whereas a spreadsheet's computational capability allows for the representation of problems involving complex variables by dealing separately with real and imaginary parts [5], [10], in what follows the discussion will be limited to a spreadsheet-based demonstration of what it means for the segment to be mapped into itself.

In exploring the behavior of orbits generated by equation (6), one has to be concerned with only those $x(t)$ that do not leave the segment $[-\pi, \pi]$ as $t$ grows larger. This gives rise to the following question: What is the range of parameter $\alpha$ that keeps orbits $x(t)$, whatever their qualitative behavior is, being bounded by $\pi$ in absolute value? With this in mind, consider the function $g(x) = x - \alpha \sin x$ for $\alpha > 2$. The use of a spreadsheet in combination with the basic techniques of differential calculus makes it possible to find those values of $\alpha$ for which the function $g(x)$ maps the segment $[-\pi, \pi]$ into itself. To this end, one can begin with constructing an interactive graphing environment allowing for the simultaneous variation of a (slider-controlled) parameter and the corresponding

| | A | B | C | D |
|---|---|---|---|---|
| 1 | | ALPHA | | |
| 2 | 408 | 4.08 | | |
| 3 | | | | |
| 4 | | | | |
| 5 | -3.14159 | -3.14159 | -3.14159 | 3.14159 |
| 6 | -3.04159 | -2.63427 | -3.14159 | 3.14159 |
| 7 | -2.94159 | -2.13102 | -3.14159 | 3.14159 |
| 8 | -2.84159 | -1.63587 | -3.14159 | 3.14159 |
| 9 | -2.74159 | -1.15277 | -3.14159 | 3.14159 |
| 10 | -2.64159 | -0.68554 | -3.14159 | 3.14159 |
| 11 | -2.54159 | -0.23785 | -3.14159 | 3.14159 |
| 12 | -2.44159 | 0.18682 | -3.14159 | 3.14159 |
| 13 | -2.34159 | 0.58522 | -3.14159 | 3.14159 |
| 14 | -2.24159 | 0.95438 | -3.14159 | 3.14159 |
| 15 | -2.14159 | 1.29161 | -3.14159 | 3.14159 |
| 16 | -2.04159 | 1.59453 | -3.14159 | 3.14159 |
| 17 | -1.94159 | 1.86113 | -3.14159 | 3.14159 |
| 18 | -1.84159 | 2.08972 | -3.14159 | 3.14159 |
| 19 | -1.74159 | 2.27904 | -3.14159 | 3.14159 |
| 20 | -1.64159 | 2.42819 | -3.14159 | 3.14159 |

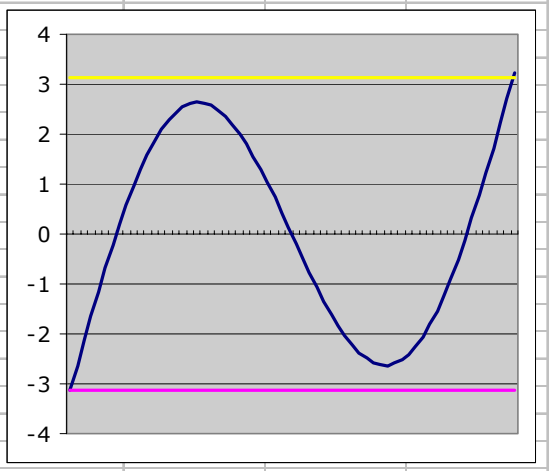



Figure 2: $g(x) = x - \alpha \sin x; |g(x)| < \pi,\ \ x \in (-\pi, \pi),\ \alpha = 4.08.$

graph of a function that depends on this parameter. By using a slider that controls the variation of $\alpha$ (Figure 2, cell `B2`), one can first locate $\alpha = 4.6$ (Figure 3) as an approximate value of the parameter such that when $\alpha > 4.6$ the inequality $|g(x)| \leq \pi$ cannot be satisfied for all $x \in [-\pi, \pi]$ (Figure 4).

One can find a more accurate value of $\alpha$ as a solution to equation (7) and, in doing so, clarify the statement of Proposition 5.1 [25, p. 149] which gives the exact estimate of mapping the segment $[-\pi, \pi]$ into itself. To this end, note that $g(-x) = -g(x)$and $g'(x) = 1 - \alpha \cos x$. Therefore, the equation $g'(x) = 0$ has two roots, $\pm \arccos(\frac{1}{\alpha})$, in the interval $(-\pi, \pi)$. The graph pictured in Figure 3 suggests that $g[-\arccos(1/\alpha)] = -g[\arccos(1/\alpha)] = \pi$; thus the value of $\alpha$ should be chosen to satisfy the equation $-\arccos(\frac{1}{\alpha}) + \alpha \sin[\arccos(\frac{1}{\alpha})] = \pi$, whence

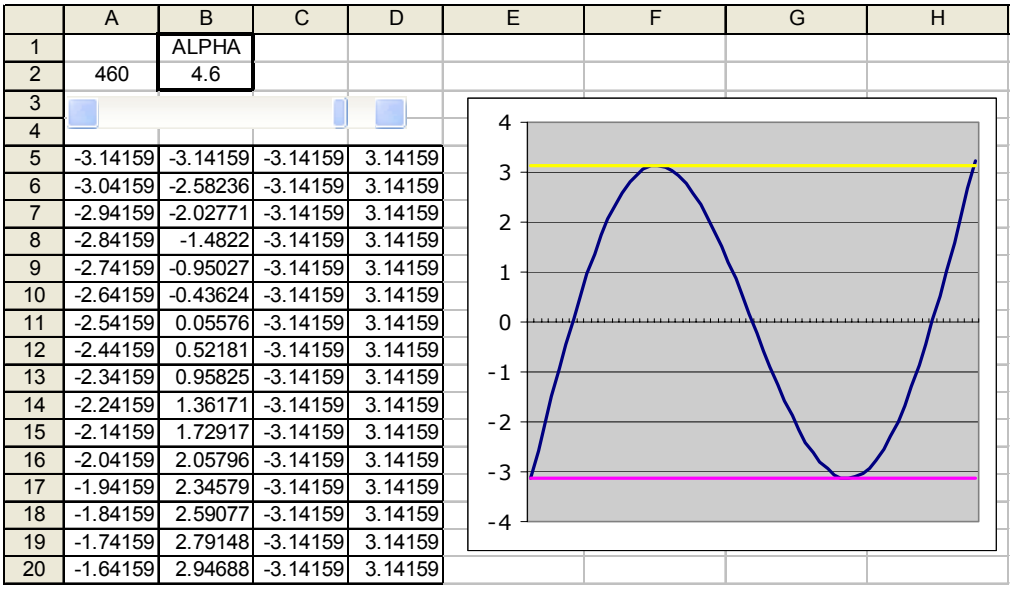
$$\sqrt{\alpha^2 - 1} = \pi + \arccos\left(\frac{1}{\alpha}\right) \tag{7}$$

Equation (7) can be solved numerically within a spreadsheet using the method of iterations as described in [29]. To this end, equation (7) can be rewritten in the form

$$\alpha = \sqrt{1 + \left(\pi + \arccos\left(\frac{1}{\alpha}\right)\right)^2}, \tag{8}$$

enabling the construction of the following recurrence

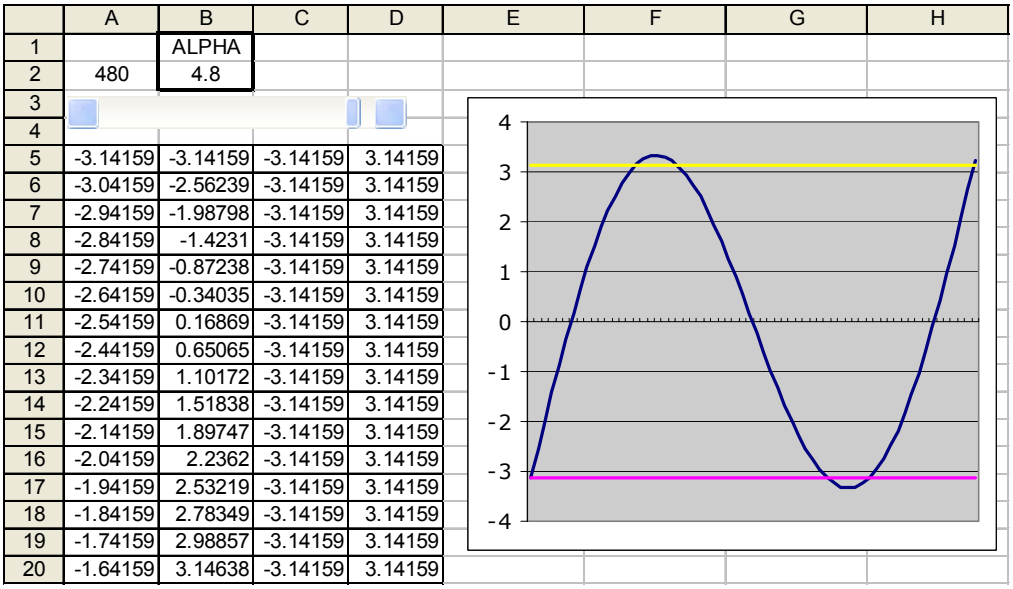
$$\alpha_{n+1} = \sqrt{1 + \left(\pi + \arccos\left(\frac{1}{\alpha_n}\right)\right)^2} \tag{9}$$



| | A | B | C | D |
|---|---|---|---|---|
| 1 | | ALPHA | | |
| 2 | 460 | 4.6 | | |
| 3 | | | | |
| 4 | | | | |
| 5 | -3.14159 | -3.14159 | -3.14159 | 3.14159 |
| 6 | -3.04159 | -2.58236 | -3.14159 | 3.14159 |
| 7 | -2.94159 | -2.02771 | -3.14159 | 3.14159 |
| 8 | -2.84159 | -1.4822 | -3.14159 | 3.14159 |
| 9 | -2.74159 | -0.95027 | -3.14159 | 3.14159 |
| 10 | -2.64159 | -0.43624 | -3.14159 | 3.14159 |
| 11 | -2.54159 | 0.05576 | -3.14159 | 3.14159 |
| 12 | -2.44159 | 0.52181 | -3.14159 | 3.14159 |
| 13 | -2.34159 | 0.95825 | -3.14159 | 3.14159 |
| 14 | -2.24159 | 1.36171 | -3.14159 | 3.14159 |
| 15 | -2.14159 | 1.72917 | -3.14159 | 3.14159 |
| 16 | -2.04159 | 2.05796 | -3.14159 | 3.14159 |
| 17 | -1.94159 | 2.34579 | -3.14159 | 3.14159 |
| 18 | -1.84159 | 2.59077 | -3.14159 | 3.14159 |
| 19 | -1.74159 | 2.79148 | -3.14159 | 3.14159 |
| 20 | -1.64159 | 2.94688 | -3.14159 | 3.14159 |



Figure 3: $g(x) = x - \alpha \sin x; \max_{x\in(-\pi,0]} g(x) = \pi; \min_{x\in[0,\pi)} g(x) = -\pi$



| | A | B | C | D |
|---|---|---|---|---|
| 1 | | ALPHA | | |
| 2 | 480 | 4.8 | | |
| 3 | | | | |
| 4 | | | | |
| 5 | -3.14159 | -3.14159 | -3.14159 | 3.14159 |
| 6 | -3.04159 | -2.56239 | -3.14159 | 3.14159 |
| 7 | -2.94159 | -1.98798 | -3.14159 | 3.14159 |
| 8 | -2.84159 | -1.4231 | -3.14159 | 3.14159 |
| 9 | -2.74159 | -0.87238 | -3.14159 | 3.14159 |
| 10 | -2.64159 | -0.34035 | -3.14159 | 3.14159 |
| 11 | -2.54159 | 0.16869 | -3.14159 | 3.14159 |
| 12 | -2.44159 | 0.65065 | -3.14159 | 3.14159 |
| 13 | -2.34159 | 1.10172 | -3.14159 | 3.14159 |
| 14 | -2.24159 | 1.51838 | -3.14159 | 3.14159 |
| 15 | -2.14159 | 1.89747 | -3.14159 | 3.14159 |
| 16 | -2.04159 | 2.2362 | -3.14159 | 3.14159 |
| 17 | -1.94159 | 2.53219 | -3.14159 | 3.14159 |
| 18 | -1.84159 | 2.78349 | -3.14159 | 3.14159 |
| 19 | -1.74159 | 2.98857 | -3.14159 | 3.14159 |
| 20 | -1.64159 | 3.14638 | -3.14159 | 3.14159 |



Figure 4: For $g(x) = x - \alpha \sin x; \alpha > 4.6, \exists x_0 : |g(x_0)| > \pi$

| | | A | B | C |
|---|---|---|---|---|
| 1 | number of iterations | alpha | | precision |
| 2 | 0 | 4.5 | | 7 |
| 3 | 1 | 4.5983475 | | |
| 4 | 2 | 4.603103 | | |
| 5 | 3 | 4.6033277 | | |
| 6 | 4 | 4.6033383 | | |
| 7 | 5 | 4.6033388 | | |
| 8 | 6 | 4.6033388 | | |

Figure 5: Finding the root of equation (7) through iterating sequence (9).

which can be iterated by using the spreadsheet's remarkable feature of recurrent counting.

The result of iterating recurrence relation (9) with $\alpha_0 = 4.5$ is shown in Figure 5. Note that if after several iterations, the equality $\alpha_{n+1} = \alpha_n$ is satisfied within a specified precision, then $\alpha_n$ is the root of equation (8) with the same precision. The very precision of iterations can be controlled by a slider. To this end, one can use the spreadsheet function `TRUNC(number, numDigits)` which, when applied to iterations of sequence (9) displayed in column `B`, truncates each iteration (number) with precision (numDigits) displayed in cell `D2` to which a slider is attached. In other words, by using a slider one can control the precision of a computed root through the formula `=TRUNC(SQRT(1+(PI()+ACOS(1/B2))^2),D$2)` defined in cell `B3` and replicated down column `B`. As Figure 5 indicates, one can see that the value $\alpha_1 = 4.6033388$ (cell `B8`) is the root of equation (7) found by the method of iterations with the precision of seven digits after the decimal point.

Moreover, the environment makes it possible to terminate the iteration process once two consecutive iterations coincide within a specified accuracy (precision). To this end, the last (action-type) formula can be replaced by the following conditional formula

`=IF(OR(B2=B1,B2=""),"",TRUNC(SQRT(1+(PI()+ACOS(1/B2))^2),D$2))`

In turn, the so refined formula can be connected to computing the number of iterations required for a root finding process to be completed. To this end, the formula `=IF(B3=" "," ",1+A2)` can be defined in cell `A3` and replicated down column `A`. This computational improvement allows one to see that the higher the precision of an iterated root, the bigger the number of iterations required for achieving that precision.

One can also use the goal seek command from the Tools Menu of Excel to find the root of equation (7) as an input value the difference between its left-hand and right-hand sides needs in order to return zero. It appears, however, that the method of iterations gives a higher precision than the goal seeking command. Furthermore, the mapping of

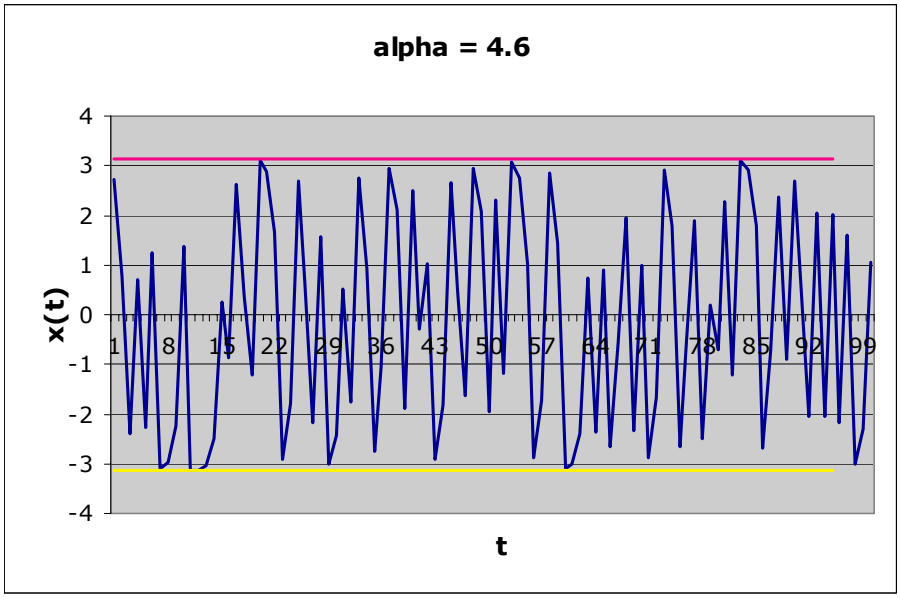



Figure 6: $\alpha = 4.6 : |x(t)| \leq \pi, t \in N$.

a segment into itself provides a context for revisiting the method of iterations—one of the earliest root-finding algorithms.

Finally, assuming $2 < \alpha \leq \alpha^*$ and choosing $x(0) \in [-\pi, \pi]$, one can use equation (6) to conclude that $x(1) = g(x(0)) \in [-\pi, \pi]$. Similarly, the inclusion $x(2) = g(x(1)) \in [-\pi, \pi]$ holds true. Continuing in this vein, one can come to the conclusion that for $2 < \alpha \leq \alpha^*$ where $\alpha^*$ is the root of equation (7), the sequence $x(t) \in [-\pi, \pi]$, $\forall t \in N$. Also, one can see that when $\alpha$ becomes greater than $\alpha^*$, the sequence $x(t)$leaves the segment $[-\pi, \pi]$ as graphs in Figures 6 and 7 indicate. Note that the described approach provides the exact estimate ($\alpha^*$) of the mapping of the segment $[-\pi, \pi]$ into itself and can be applied to other nonlinearities, like $g(x) = x - \alpha\, sign \sin x$ (see equation (10) below) for which the estimate of parameter $\alpha$ is different, of course. A slight modification of the environment pictured in Figure 2 would result in $\pi$ being that estimate.

## 4 The bifurcation diagram as a tool for refining bifurcation parameters

The bifurcation diagram (sometimes called the *orbit diagram* or *Feigenbaum plot*) of a dynamic system represents a set of attracting orbits that follow a transient behavior of an orbit. A simple example known as "the $3x+1$ problem" [23] can be used to clarify the point. Let $x(t+1) = 3x(t)+1$ if $x(t)$ is odd, and $x(t+1) = x(t)/2$ if $x(t)$ is even, $t \in N$. The sequence $5, 16, 8, 4, 2, 1, 4, 2, 1, 4, 2, 1, \ldots$ is the orbit of the point $x(0) = 5$ under this iteration. That kind of sequence, attracted by the 3-cycle $(4, 2, 1)$, is sometimes called

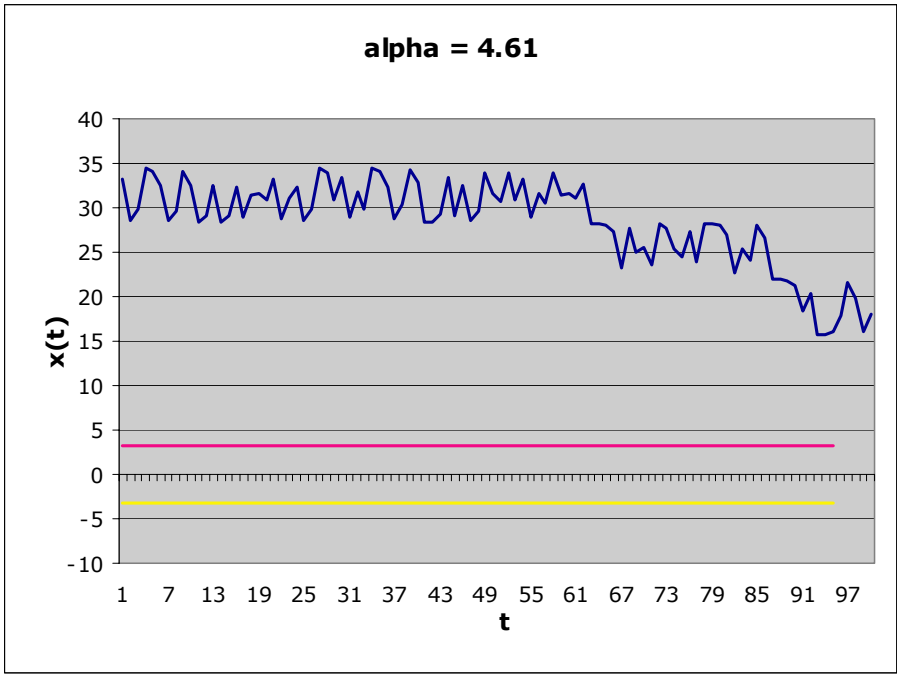



Figure 7: $\alpha = 4.61 : |x(t)| > \pi, t \in N, t > N$.

an *eventually periodic sequence*, the first three terms of which, viz., 5, 16, 8, characterize its transient behavior. Here, the initial value $x(0)$ may be considered as an integer parameter, on which the iteration depends. The attractors represent the set of values (called a cycle), which the iteration converges to; in other words, it shows what happens with iterations of $x(0)$ after they have settled down. In that way, a bifurcation diagram can be used to demonstrate the limiting behavior of iterations.

In the case of equation (6), one can use a spreadsheet to construct a bifurcation diagram which relates any value of parameter $\alpha$ to the corresponding set of attracting orbits. This diagram can then be zoomed-in (e.g., by changing the maximum and minimum values of the $\alpha$-axis scale using the *Format Axis* feature) to locate approximate values of parameter $\alpha$ for which different bifurcations of orbits occur. Once such values are known, one can refine them by iterating equation (6) within the appropriate range of the parameter using spreadsheet techniques described in the next section.

In order to explain the bifurcation diagrams of Figures 8, 9 and 10, note that a cycle is considered to be globally stable if the change in initial data does not change the cycle's qualitative behavior. However, when the limiting cycle that attracts an orbit depends on initial value, such a cycle is considered to be locally stable. As shown in Figure 8, the diagram does not depend on the value of $x(0)$ until $\alpha$ reaches $\pi$ – at that point both the upper and lower branches split into two branches each of which corresponds to either $x(0) > 0$ or $x(0) < 0$. Figures 9 and 10 show two separate bifurcation diagrams

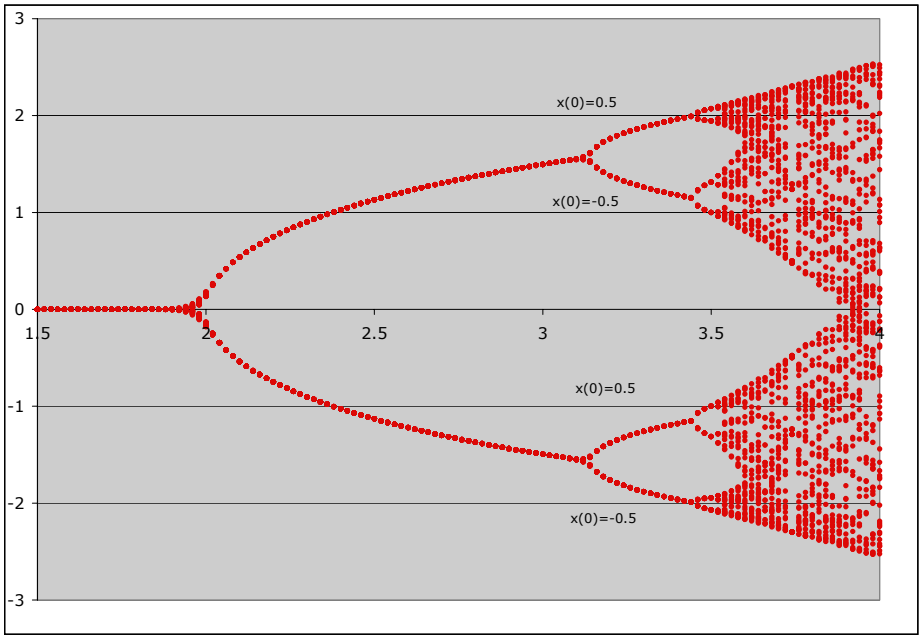



Figure 8: Bifurcation diagram for equation (6).

constructed for $x(0) = 0.5$ and $x(0) = -0.5$, respectively. One can see how the symmetry of cycles of the mapping of the segment into itself disappears as one attempts to construct a bifurcation diagram for a specific value of $x(0)$ – the initial values $\pm 0.5$ are used to demonstrate this phenomenon. It is interesting to note that in comparison with well-known orbit diagrams for quadratic maps (e.g., equation (3); for their spreadsheet-based representation, see [11], [21], [36]), the bifurcation diagram for equation (6) exhibits a qualitatively different type of orbital behavior; namely, it shows the emergence of two locally stable non-symmetrical 2-cycles from one globally stable symmetrical 2-cycle as the parameter $\alpha$ approaches $\pi$. It appears that such qualitative difference is due to the difference in the number of points of extremum of the two systems' non-linear characteristics [38].

A computational part of the spreadsheet that generates the bifurcation diagram is shown in Figure 11. The spreadsheet formulas used to construct the environment are as

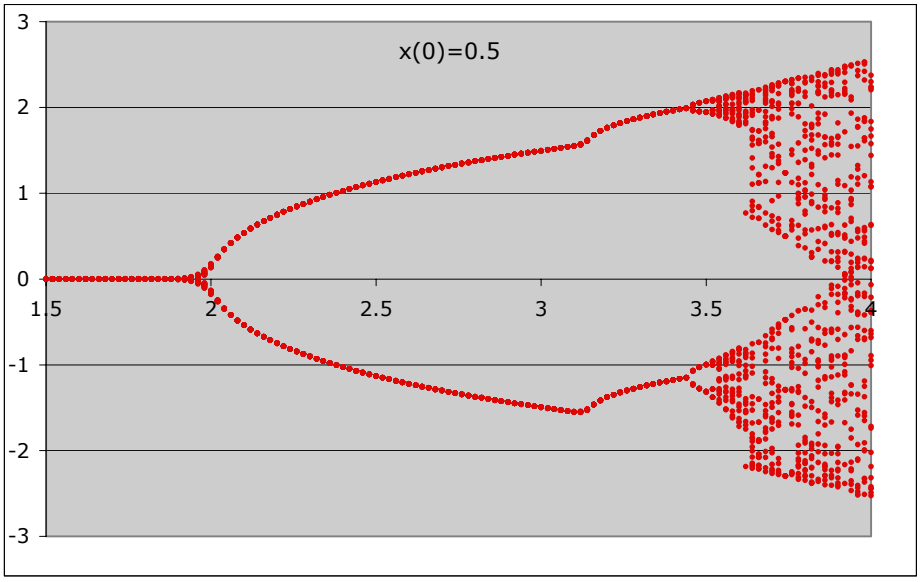



Figure 9: Bifurcation diagram for $x(0) = 0.5$.

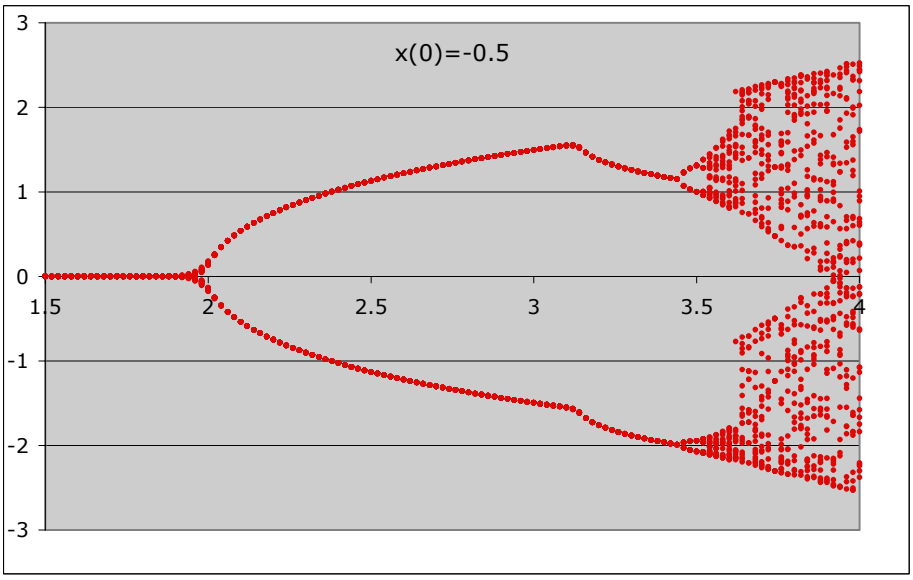



Figure 10: Bifurcation diagram for $x(0) = -0.5$.

| | A | B | C | D | E | F | G | H | I | J | K | L | M | N | O | P |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | | its | parameter | attractors | iterations | attractors | | iters count | attractors | iterations | attractors | 40 | 0.02 | 50 | 0.5 | 0 |
| 2 | 1 | 1 | 1.5 | 7.722E-13 | -0.5 | 7.722E-13 | | 1 | -7.72E-13 | 0.5 | -7.72E-13 | | | | initial value | change sign of |
| 3 | 2 | 2 | 1.5 | -3.86E-13 | 0.2191383 | -3.86E-13 | | 2 | 3.861E-13 | -0.219138 | 3.861E-13 | | | | (abs) | initial value |
| 4 | 3 | 3 | 1.5 | 1.931E-13 | -0.106945 | 1.931E-13 | | 3 | -1.93E-13 | 0.1069446 | -1.93E-13 | number of | increment | | | |
| 5 | 4 | 4 | 1.5 | -9.65E-14 | 0.0531667 | -9.65E-14 | | 4 | 9.653E-14 | -0.053167 | 9.653E-14 | iterations | | | | |
| 6 | 5 | 5 | 1.5 | 4.826E-14 | -0.026546 | 4.826E-14 | | 5 | -4.83E-14 | 0.0265458 | -4.83E-14 | | | | | |

Figure 11: Numerical part of the bifurcation diagram.

follows.

```
(F2)=IF(B2=1,OFFSET(B2,L$1-1,3),F1-C2*SIN(F1))
(E2)=IF(P1=1,O1,-O1)
(E3)=IF(B3=1,$E$2,E2-C2*SIN(E2))
(D2)=F2
(B3)=IF(B2=L$1,1,B2+1)
(C3)=IF(B3=1,C2+M$1,C2)
```

In addition, cells `A2` and `B2` are entered with the numbers 1 and 1.5, respectively. The first of the above listed spreadsheet formulas includes a spreadsheet function `OFFSET` consisting of three parts (arguments): a cell, a vertical offset, and a horizontal offset. For example, the formula `=OFFSET(A1,3,5)` returns the value of cell F4 which is located three rows down and five columns to the right of cell A1. Note that the `OFFSET` function is a generalization of another spreadsheet function, `INDEX`, which use is described in [2]. Indeed, whereas the latter function makes it possible to locate the value of a designated cell within a fixed range (array) only, the former function makes it possible to work with evolving arrays. In particular, the use of the `OFFSET` function in the construction of bifurcation diagrams makes it possible to control the number of iterations required for a transient process to be completed.

## 5 The use of conditional formatting in demonstrating cyclic behavior of orbits

The behavior of the orbit generated by equation (6) for a particular value of the parameter $\alpha$ can be demonstrated both numerically and graphically. As mentioned above, by using the option of zooming-in at a bifurcation diagram which shows the limiting behavior of the orbits as the function of $\alpha$, a specific value of the parameter for which such representations are sought can be selected. For example, Figures 12 and 13 show graphically how the global asymptotic stability turns into oscillations as the parameter $\alpha$ leaves the interval $(0, 2)$. Although $t$ is discrete, we have chosen a spreadsheet chart-type in which the points are joined by lines in order to clearly emphasize the oscillations. These oscillations appear to be periodic and this suggests that conditional formatting (CF) could complement a chart-type representation of the cyclic behavior of orbits by being applied to the numerical representation of orbits that, otherwise, are difficult to

understand visually. As noted in [4], CF is a relatively recent feature of the modern graphical spreadsheet allowing for automatic formatting of any cell based on its current value. To this end, the following environment can be constructed (see Figure 14).

First, cell `B2` – the value of parameter $\alpha$ – is entered with the formula `=A2/100` and it is controlled through a slider attached to cell `A2` the content of which varies in the range [100, 500] enabling the variation of $\alpha$ within the range $[1, 5]$ with step 0.01. Cell `C2` – the value of $x(0)$ – is entered with the formula `=IF(E2=1,D2/100,-D2/100)` enabling the change of sign and absolute value of the initial value through the sliders attached to cells `D2` and `E2` respectively.

Second, column `B`, beginning from cell `B6`, includes orbits $x(t)$ with $x(0)$ duplicated in cell `B6`. Setting 1000 iterations as the orbital transient period, cell `D7` is entered with the formula `=TRUNC(B1005,I$1)` which is replicated down, enabling one to consider the iterations of $x(0)$ when they have settled down. In addition, through the slider attached to cell `I1` one can control the precision of the truncation.

Third, column `E`, beginning from cell `E7` contains counting numbers that serve as mediators in identifying periodic behavior of the iterations. To this end, in cell `F8` the formula `=IF(D$7=D8,E8-E$7,“ “)` is defined and replicated down column `F`, enabling the spreadsheet to identify the orbits' attractor. Furthermore, the smallest number in this column represents the period sought. The result of exploration for a particular value of parameter $\alpha$ is displayed in the range `F5:I5` as follows: cell `I5` contains the formula `=IF(SUM(F7:F73)=0, “INSTABILITY”, MIN(F7:F73))`; cell `H5` contains the formula `=IF(I5=”INSTABILITY”, “ “, “Period=”)`.

Finally, CF of iterations can be used by highlighting the range `D7:D73`, opening the dialogue box of CF and defining two conditions – cell value is equal to `=$D$7`, and cell value is equal to `=$D$8` – depending on which a cell in this range may be colored (formatted) in one of two chosen colors. Concurrently, the behavior of iterations can be shown graphically using the chart-wizard feature of a spreadsheet. As Figure 14 illustrates, for $\alpha = 2.1$ the orbits behave as a two-cycle and all cells containing the settled down iterations (column `D`) have been formatted to demonstrate this type of behavior.

Also, by changing the values of $\alpha$ and $x(0)$, global asymptotic stability of cycles can be demonstrated in the CF setting. However, as parameter $\alpha$ approaches one of its bifurcation values, the corresponding cycle loses its stability – a phenomenon represented through a non-regular behavior of iterations and the disappearance of regular coloring patterns. As parameter $\alpha$ changes further, a new globally asymptotically stable cycle emerges with period being the double of a previous one. Thus, by changing the value of parameter $\alpha$ one can observe the phenomenon of period-doubling bifurcations using a combination of spreadsheet graphing and CF applied to numerical iterations (Figures 15-18). In particular, one can observe (Figures 19, 20) in an alternative environment how in the neighborhood of $\alpha = \pi$ a globally stable 2-cycle is replaced by two locally stable 2-cycles – a characteristic property of orbits generated by equation (6).

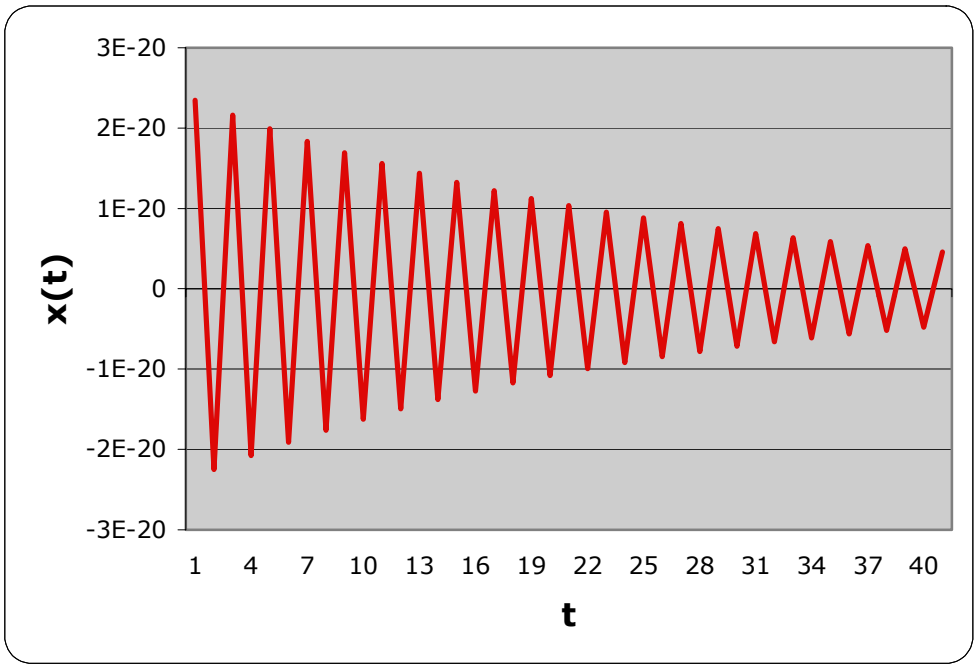



Figure 12: Approaching the first bifurcation: asymptotic stability with $\alpha = 1.96$

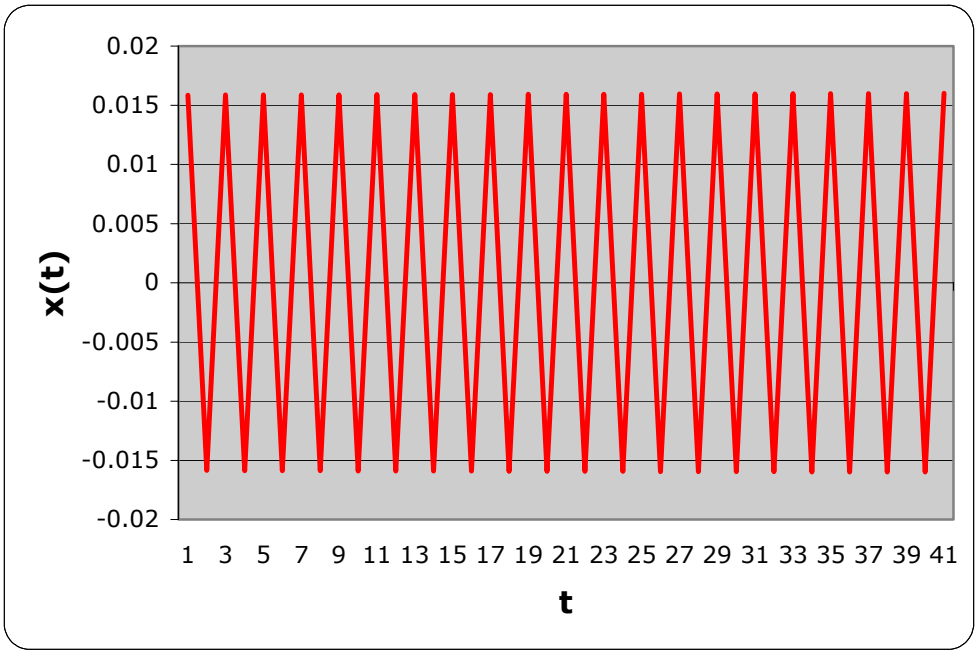



Figure 13: The first bifurcation: oscillations with $\alpha = 2.00$

DISCRETE PHASE-LOCKED LOOP SYSTEMS

| | | A | B | C | D | E | F | G | H | I |
|---|---|---|---|---|---|---|---|---|---|---|
| | change alpha | alpha | x(0) | change x(0) | change sign of x(0) | | | Truncate to | 7 | digits |
| 1 | | 3.141592654 | -0.07 | | | | | | | |
| 2 | | | | | | | | | | |
| 3 | | | | | | | | | | |
| 4 | | | | | | ALPHA= | 3.1416 | | INSTABILITY | |
| 5 | | -0.07 | | | | | | | | |
| 6 | | 0.14973193536675 | | 1.5848265000 | | | | | | |
| 7 | | -0.31890909347347 | | -1.5564568000 | | | | | | |
| 8 | | 0.66607711065742 | | 1.5848127000 | | | | | | |
| 9 | | -1.27513300531516 | | -1.5564712000 | | | | | | |
| 10 | | 1.73014304602607 | | 1.5847990000 | | | | | | |
| 11 | | -1.37164924846873 | | -1.5564856000 | | | | | | |
| 12 | | 1.70785193342589 | | 1.5847853000 | | | | | | |
| 13 | | -1.40428066479432 | | -1.5564999000 | | | | | | |
| 14 | | 1.69385833186984 | | 1.5847716000 | | | | | | |
| 15 | | -1.42397578480648 | | -1.5565141000 | | | | | | |
| 16 | | 1.68381713845756 | | 1.5847580000 | | | | | | |
| 17 | | -1.43773197750040 | | -1.5565283000 | | | | | | |
| 18 | | 1.67608897989823 | | 1.5847445000 | | | | | | |
| 19 | | -1.448105056 | | -1.5565425000 | | | | | | |
| 20 | | 1.669871815 | | 1.5847309000 | | | | | | |
| 21 | | -1.456314586 | | -1.5565566000 | | | | | | |
| 22 | | 1.664713578 | | 1.5847174000 | | | | | | |
| 23 | | -1.463034127 | | -1.5565707000 | | | | | | |
| 24 | | 1.660334999 | | 1.5847040000 | 19 | | | | | |

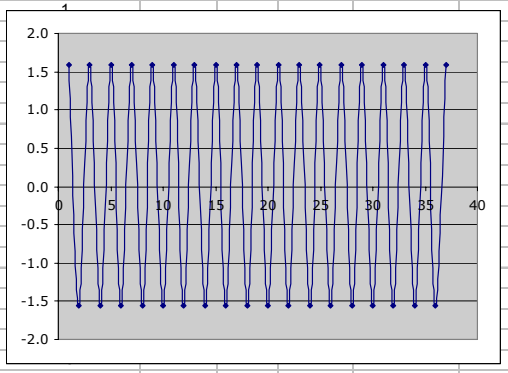



Figure 14: CF enhances spreadsheet graphics.

| | | A | B | C | D | E | F | G | H | I |
|---|---|---|---|---|---|---|---|---|---|---|
| | change alpha | alpha | x(0) | change x(0) | change sign of x(0) | | | Truncate to | 7 | digits |
| 1 | | 2.6 | -0.07 | | | | | | | |
| 2 | | | | | | | | | | |
| 3 | | | | | | | | | | |
| 4 | | | | | | ALPHA= | 2.6 | Period= | 2 | |
| 5 | | -0.07 | | | | | | | | |
| 6 | | 0.11185140307759 | | 1.2214962000 | | | | | | |
| 7 | | -0.17835624197835 | | -1.2214962000 | | | | | | |
| 8 | | 0.28291529947562 | | 1.2214962000 | | | | | | |
| 9 | | -0.44289091059432 | | -1.2214962000 | | | | | | |
| 10 | | 0.67134750749559 | | 1.2214962000 | | | | | | |
| 11 | | -0.94596072660732 | | -1.2214962000 | | | | | | |
| 12 | | 1.16279344930453 | | 1.2214962000 | | | | | | |
| 13 | | -1.22378571964883 | | -1.2214962000 | | | | | | |
| 14 | | 1.22123756297816 | | 1.2214962000 | | | | | | |
| 15 | | -1.22152462985213 | | -1.2214962000 | | | | | | |
| 16 | | 1.22149308290233 | | 1.2214962000 | | | | | | |
| 17 | | -1.22149655957138 | | -1.2214962000 | | | | | | |
| 18 | | 1.22149617654015 | | 1.2214962000 | | | | | | |
| 19 | | -1.221496219 | | -1.2214962000 | | | | | | |
| 20 | | 1.221496214 | | 1.2214962000 | | | | | | |
| 21 | | -1.221496215 | | -1.2214962000 | | | | | | |
| 22 | | 1.221496215 | | 1.2214962000 | | | | | | |
| 23 | | -1.221496215 | | -1.2214962000 | | | | | | |
| 24 | | 1.221496215 | | 1.2214962000 | 19 | 18 | | | | |

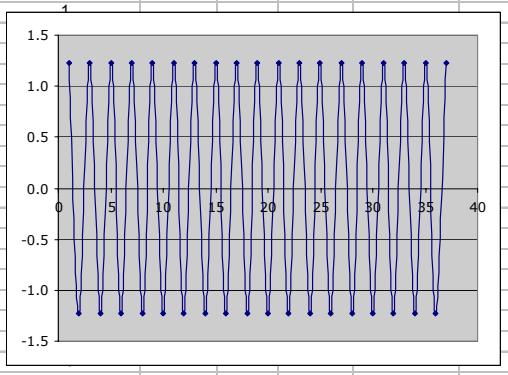



Figure 15: CF shows a 2-cycle.

| | | A | B | C | D | E | F | G | H | I |
|---|---|---|---|---|---|---|---|---|---|---|
| | change alpha | alpha | x(0) | change x(0) | change sign of x(0) | | | Truncate to | 7 | digits |
| 1 | | 3.48 | -0.07 | | | | | | | |
| 2 | | | | | | | | | | |
| 3 | | | | | | | | | | |
| 4 | | | | | | ALPHA= | 3.48 | Period= | 4 | |
| 5 | | -0.07 | | | | | | | | |
| 6 | | 0.17340110873461 | | 1.0278513000 | | | | | | |
| 7 | | -0.42701528004975 | | -1.9516927000 | | | | | | |
| 8 | | 1.01424739600048 | | 1.2789017000 | | | | | | |
| 9 | | -1.94056221780164 | | -2.0538956000 | | | | | | |
| 10 | | 1.30423146383203 | | 1.0278513000 | | | | | | |
| 11 | | -2.05286004229581 | | -1.9516927000 | | | | | | |
| 12 | | 1.03055934490344 | | 1.2789017000 | | | | | | |
| 13 | | -1.95384277603131 | | -2.0538956000 | | | | | | |
| 14 | | 1.27396279863207 | | 1.0278513000 | | | | | | |
| 15 | | -2.05384794343148 | | -1.9516927000 | | | | | | |
| 16 | | 1.02797610093804 | | 1.2789017000 | | | | | | |
| 17 | | -1.95179231766849 | | -2.0538956000 | | | | | | |
| 18 | | 1.27867338316430 | | 1.0278513000 | | | | | | |
| 19 | | -2.05389522 | | -1.9516927000 | | | | | | |
| 20 | | 1.027852402 | | 1.2789017000 | | | | | | |
| 21 | | -1.951693632 | | -2.0538956000 | | | | | | |
| 22 | | 1.278899754 | | 1.0278513000 | | | | | | |
| 23 | | -2.053895631 | | -1.9516927000 | | | | | | |
| 24 | | 1.027851328 | | 1.2789017000 | 19 | | | | | |

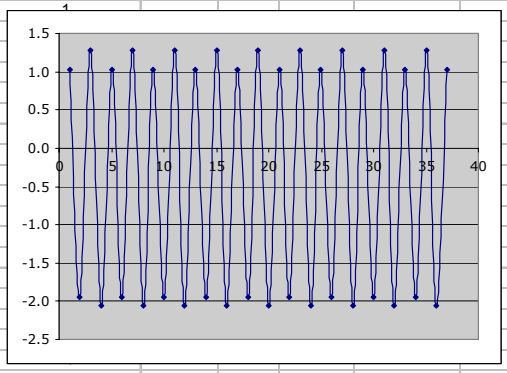



Figure 16: CF shows a 4-cycle.

| | | A | B | C | D | E | F | G | H | I |
|---|---|---|---|---|---|---|---|---|---|---|
| | change alpha | alpha | x(0) | change x(0) | change sign of x(0) | | | Truncate to | 7 | digits |
| 1 | | 3.52 | -0.07 | | | | | | | |
| 2 | | | | | | | | | | |
| 3 | | | | | | | | | | |
| 4 | | | | | | ALPHA= | 3.52 | Period= | 8 | |
| 5 | | -0.07 | | | | | | | | |
| 6 | | 0.17619882262812 | | 1.0036576000 | | | | | | |
| 7 | | -0.44081678416170 | | -1.9652567000 | | | | | | |
| 8 | | 1.06109086609767 | | 1.2844216000 | | | | | | |
| 9 | | -2.01147579915066 | | -2.0922236000 | | | | | | |
| 10 | | 1.17223058269271 | | 0.9600002000 | | | | | | |
| 11 | | -2.07186679079326 | | -1.9235545000 | | | | | | |
| 12 | | 1.01541556614013 | | 1.3796956000 | | | | | | |
| 13 | | -1.97552751651249 | | -2.0762254000 | | | | | | |
| 14 | | 1.26008562997474 | | 1.0036576000 | | | | | | |
| 15 | | -2.09136453870823 | | -1.9652567000 | | | | | | |
| 16 | | 0.96236466776662 | | 1.2844216000 | | | | | | |
| 17 | | -1.92595535475978 | | -2.0922236000 | | | | | | |
| 18 | | 1.37436567055916 | | 0.9600002000 | | | | | | |
| 19 | | -2.077942802 | | -1.9235545000 | | | | | | |
| 20 | | 0.999008803 | | 1.3796956000 | | | | | | |
| 21 | | -1.961082487 | | -2.0762254000 | | | | | | |
| 22 | | 1.294214314 | | 1.0036576000 | | | | | | |
| 23 | | -2.092005984 | | -1.9652567000 | | | | | | |
| 24 | | 0.960599583 | | 1.2844216000 | 19 | | | | | |

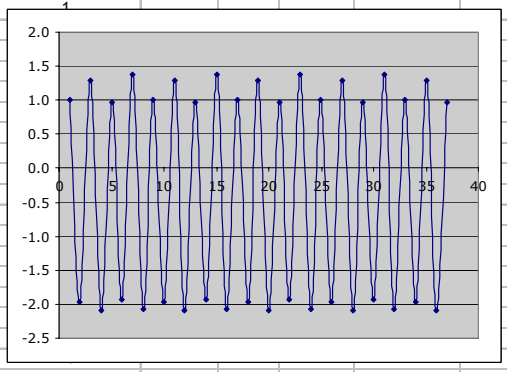



Figure 17: CF shows an 8-cycle.

| | | A | B | C | D | E | F | G | H | I |
|---|---|---|---|---|---|---|---|---|---|---|
| | change alpha | alpha | x(0) | change x(0) | change sign of x(0) | | | Truncate to | 7 | digits |
| 1 | | 3.53 | -0.07 | | | | | | | |
| 2 | | | | | | | | | | |
| 3 | | | | | | | | | | |
| 4 | | | | | | **ALPHA=** | **3.53** | **Period=** | **16** | |
| 5 | | -0.07 | | | | | | | | |
| 6 | | 0.17689825110149 | | 0.9511121000 | | | | | | |
| 7 | | -0.44430084490378 | | -1.9225264000 | | | | | | |
| 8 | | 1.07298744244804 | | 1.3913600000 | | | | | | |
| 9 | | -2.02857974476288 | | -2.0819638000 | | | | | | |
| 10 | | 1.13795148552944 | | 0.9968101000 | | | | | | |
| 11 | | -2.06649829513266 | | -1.9674833000 | | | | | | |
| 12 | | 1.03861338151837 | | 1.2883981000 | | | | | | |
| 13 | | -2.00319278649689 | | -2.1017782000 | | | | | | |
| 14 | | 1.20192057987009 | | 0.9421768000 | | | | | | |
| 15 | | -2.09062798175547 | | -1.9130285000 | | | | | | |
| 16 | | 0.97306895408834 | | 1.4122592000 | | | | | | |
| 17 | | -1.94488801223541 | | -2.0734720000 | | | | | | |
| 18 | | 1.34097693602852 | | 1.0198549000 | | | | | | |
| 19 | | -2.096210729 | | -1.9878183000 | | | | | | |
| 20 | | 0.957649327 | | 1.2396578000 | | | | | | |
| 21 | | -1.929329927 | | -2.0985671000 | | | | | | |
| 22 | | 1.376205818 | | 0.9511121000 | | | | | | |
| 23 | | -2.087172255 | | -1.9225264000 | | | | | | |
| 24 | | 0.982565892 | | 1.3913600000 | 19 | | | | | |

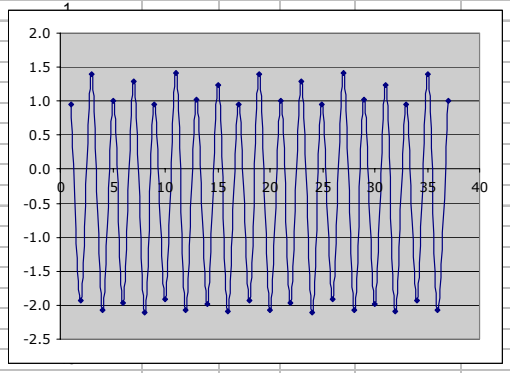



Figure 18: CF shows a 16-cycle.

| | | A | B | C | D | E | F | G | H | I |
|---|---|---|---|---|---|---|---|---|---|---|
| | change alpha | alpha | x(0) | change x(0) | change sign of x(0) | | | Truncate to | 7 | digits |
| 1 | | 3.14 | 0.07 | | | | | | | |
| 2 | | | | | | | | | | |
| 3 | | | | | | | | | | |
| 4 | | | | | | **ALPHA=** | **3.14** | | **INSTABILITY** | |
| 5 | | 0.07 | | | | | | | | |
| 6 | | -0.14962054063985 | | -1.5725747000 | | | | | | |
| 7 | | 0.31843703846966 | | 1.5674202000 | | | | | | |
| 8 | | -0.66464219540646 | | -1.5725618000 | | | | | | |
| 9 | | 1.27203925128800 | | 1.5674332000 | | | | | | |
| 10 | | -1.72886835837745 | | -1.5725489000 | | | | | | |
| 11 | | 1.37198403345539 | | 1.5674461000 | | | | | | |
| 12 | | -1.70616376740957 | | -1.5725362000 | | | | | | |
| 13 | | 1.40511091714271 | | 1.5674590000 | | | | | | |
| 14 | | -1.69188848950744 | | -1.5725234000 | | | | | | |
| 15 | | 1.42511822794014 | | 1.5674718000 | | | | | | |
| 16 | | -1.68162194450980 | | -1.5725107000 | | | | | | |
| 17 | | 1.43911454577563 | | 1.5674845000 | | | | | | |
| 18 | | -1.67370082671913 | | -1.5724981000 | | | | | | |
| 19 | | 1.449688581 | | 1.5674972000 | | | | | | |
| 20 | | -1.667312225 | | -1.5724856000 | | | | | | |
| 21 | | 1.458074074 | | 1.5675098000 | | | | | | |
| 22 | | -1.661998139 | | -1.5724731000 | | | | | | |
| 23 | | 1.46495201 | | 1.5675224000 | | | | | | |
| 24 | | -1.657475664 | | -1.5724607000 | 19 | | | | | |

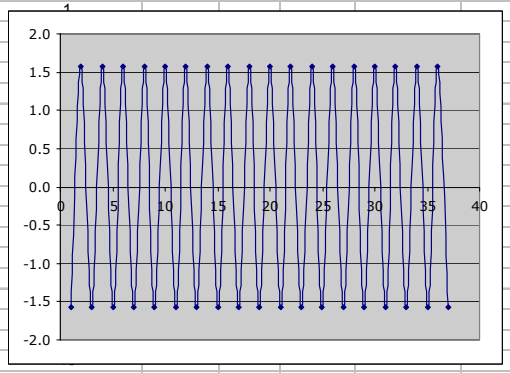
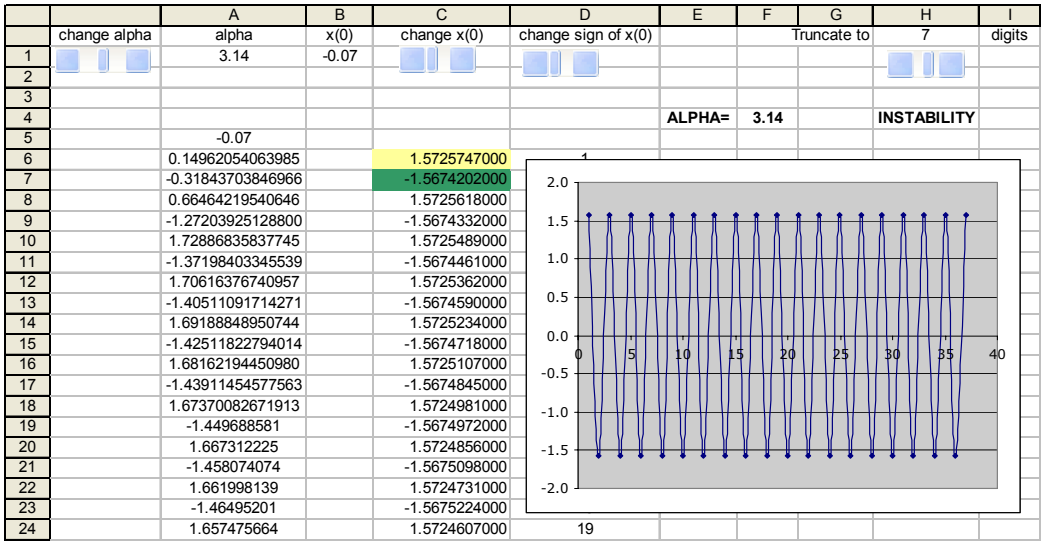



Figure 19: CF shows the loss of global stability for $\alpha = \pi : x(0) = 0.07$.



| | | A | B | C | D | E | F | G | H | I |
|---|---|---|---|---|---|---|---|---|---|---|
| | change alpha | alpha | x(0) | change x(0) | change sign of x(0) | | | Truncate to | 7 | digits |
| 1 | | 3.14 | -0.07 | | | | | | | |
| 2 | | | | | | | | | | |
| 3 | | | | | | | | | | |
| 4 | | | | | | ALPHA= | 3.14 | | INSTABILITY | |
| 5 | | -0.07 | | | | | | | | |
| 6 | | 0.14962054063985 | | 1.5725747000 | | | | | | |
| 7 | | -0.31843703846966 | | -1.5674202000 | | | | | | |
| 8 | | 0.66464219540646 | | 1.5725618000 | | | | | | |
| 9 | | -1.27203925128800 | | -1.5674332000 | | | | | | |
| 10 | | 1.72886835837745 | | 1.5725489000 | | | | | | |
| 11 | | -1.37198403345539 | | -1.5674461000 | | | | | | |
| 12 | | 1.70616376740957 | | 1.5725362000 | | | | | | |
| 13 | | -1.40511091714271 | | -1.5674590000 | | | | | | |
| 14 | | 1.69188848950744 | | 1.5725234000 | | | | | | |
| 15 | | -1.42511822794014 | | -1.5674718000 | | | | | | |
| 16 | | 1.68162194450980 | | 1.5725107000 | | | | | | |
| 17 | | -1.43911454577563 | | -1.5674845000 | | | | | | |
| 18 | | 1.67370082671913 | | 1.5724981000 | | | | | | |
| 19 | | -1.449688581 | | -1.5674972000 | | | | | | |
| 20 | | 1.667312225 | | 1.5724856000 | | | | | | |
| 21 | | -1.458074074 | | -1.5675098000 | | | | | | |
| 22 | | 1.661998139 | | 1.5724731000 | | | | | | |
| 23 | | -1.46495201 | | -1.5675224000 | | | | | | |
| 24 | | 1.657475664 | | 1.5724607000 | 19 | | | | | |



Figure 20: CF shows the loss of global stability for $\alpha = \pi : x(0) = -0.07$.

## 6 Amplitude of a cycle as a function of the parameter $\alpha$

As mentioned by Durkin [12], using computer experimentation in exploring dynamic systems with a complex behavior of orbits often results in experimental findings that stimulate pure mathematical investigations seeking to determine whether those findings are accurate. Leonov and Seledzhi [27] used a combination of computational experiments and formal mathematical explorations to reveal hidden behavior of cycles generated by equation (6). In educational settings, a spreadsheet can be used as a medium for experimental findings that can motivate students' mathematical explorations. In other words, through appropriately designed computational experiments, a spreadsheet can become an agent of mathematical activities for students [2]. Furthermore, as will be shown below, once a mathematical model is constructed, it can be explored within a spreadsheet, thus providing an alternative representation of the concepts involved. One such spreadsheet-based experiment can deal with the study of the dependence of the amplitude of a cycle on the parameter $\alpha$. The term *amplitude* is used here to refer to the maximum offset of a function from its baseline level, which is equal to zero in the case of the cycles studied in this paper. Therefore, one can use the spreadsheet formula `=MAX(MAX(range), ABS(MIN(range)))` for the determination of the amplitude of a cycle for each value of parameter $\alpha$. Here, the argument of the functions `MIN` and `MAX` refers the range of cells that includes the values of $x(t)$ for sufficiently large values of $t$.

To begin, consider equation (6). The environment pictured in Figure 21 is designed to construct the graph of $A(\alpha)$ and it includes three distinct parts. First, in column `B`, beginning from cell `B7`, the sequence $x(t)$ is generated through the formula `=B6-B$2*SIN(B6)`,

where cell `B6` contains the initial value $x(0)$ (controlled by a slider attached to cell `C2`) and cell `B2` contains a slider-controlled (via cell `A2`) value of parameter $\alpha$. For each value of $\alpha$, the amplitude of a corresponding cycle is calculated in cell `F3` through the formula `=MAX(MAX(B900:B1000), ABS(MIN(B900:B1000)))` which refers to a range of cycle's values that includes sufficiently many iterations.

Second, the numbers in column `D`, beginning from cell `D6`, represent values of $\alpha$ increased by the factor of 100. Column `E`, beginning from cell `E7`, contain values of the function $A(\alpha)$; that is, the amplitudes of the corresponding cycles. A powerful feature of a spreadsheet is its computational and operational functionality of interactive dynamic construction of the table representation of a function defined neither through a closed formula nor through recursion [1]. The function $A(\alpha)$ which relates parameter $\alpha$ to the amplitude $A$ of the corresponding cycle, is an example of such a function. The basic computational idea that enables its table representation is the use of a circular reference in a spreadsheet formula (that is, a reference to a cell in which the formula is defined). Indeed, the use of a circular reference in the spreadsheet formula `=IF($A$2=200," ",IF($A$2=D6,F$3,E6))` defined in cell `E6` and replicated down column `E` makes it possible to keep the values of already computed amplitudes unchanged as parameter $\alpha$ varies in cell `B2` controlled by a slider attached to cell `A2`. This can be done through a simple relationship between the cells defining the formula `=A2/100` in cell `B2`.

Finally, the third part of this environment includes an Excel chart that represents the graph of the function $A(\alpha)$ constructed over the range $[2.01, 4.01]$. This graph shows that the amplitude of an orbit $x(t)$ monotonically increases on this interval, uniformly about $x(0) \in (-\pi, \pi)$. The only peculiar behavior of $A(\alpha)$ that can be discovered through spreadsheet graphing is the existence of the point in which the graph appears to lose its smoothness as $\alpha$ changes from $\alpha = 3.14$ to $\alpha = 3.15$ (Figure 21). Alternatively, this behavior can be demonstrated by graphing the difference $\Delta A = A(\alpha + \Delta\alpha) - A(\alpha)$.

A much more interesting aspect of the behavior of the function $A(\alpha)$, that may stimulate significant mathematical work on the part of students, can be observed in the case of the following equation with a discontinuous non-linearity

$$x(t+1) = x(t) - \alpha sign(sin(x(t))) \qquad \alpha > 0 \qquad t \in N \tag{10}$$

typically used in floating phase-locked loops that harness both the effects of sampling and quantizing [25], [28], [8]. To this end, the environment described in this section can be retrofitted by entering cell `B7` with the formula `=B6-$B$2*SIGN(SIN(B6))` and replicating it down column `B` thus allowing one to explore the behavior of the amplitude $A(\alpha)$ of the cycles generated by equation (10). In doing so, one can discover that the behavior of the function $A(\alpha)$ strongly depends on the value of $x(0)$. For example, Figures 22 and 23 show the graphs of $A(\alpha)$ in the case of $x(0) = 1.1$ and $x(0) = 1.5$, respectively. This kind of behavior changes, however, as $x(0)$ becomes greater than 2. For example, when $x(0) = 2.1$ and $x(0) = 2.6$, the corresponding graphs of $A(\alpha)$ have the form shown in Figures 24 and 25 respectively.

Computational experiments which enable the dynamic construction of the function

| | A | B | C | D |
|---|---|---|---|---|
| 1 | alpha | sigma(0) | | amplitude |
| 2 | 4 | 2 | 1 | |
| 3 | 400 | 200 | | 2.554867236 |
| 4 | | | | |
| 5 | sigma | | | |
| 6 | 2 | 200 | | |
| 7 | -1.637189707 | 201 | 0.172902866 | 0.071193829 |
| 8 | 2.353997369 | 202 | 0.244096696 | 0.054342152 |
| 9 | -0.480637322 | 203 | 0.298438848 | 0.045574771 |
| 10 | 1.36874021 | 204 | 0.344013619 | 0.039945173 |
| 11 | -2.549883866 | 205 | 0.383958792 | 0.035928032 |
| 12 | -0.318763424 | 206 | 0.419886824 | 0.032870679 |
| 13 | 0.93480665 | 207 | 0.452757504 | 0.030439938 |
| 14 | -2.283130343 | 208 | 0.483197442 | 0.028445326 |
| 15 | 0.744223342 | 209 | 0.511642768 | 0.026768905 |
| 16 | -1.965379434 | 210 | 0.538411672 | 0.025333188 |
| 17 | 1.727248199 | 211 | 0.563744861 | 0.024084864 |
| 18 | -2.223897199 | 212 | 0.587829725 | 0.022985873 |
| 19 | 0.952916275 | 213 | 0.610815598 | 0.022008217 |
| 20 | -2.30751729 | 214 | 0.632823815 | 0.021130769 |
| 21 | 0.655185108 | 215 | 0.653954584 | 0.020337244 |
| 22 | -1.782039022 | 216 | 0.674291829 | 0.01961486 |
| 23 | 2.129045408 | 217 | 0.693906688 | 0.018953418 |

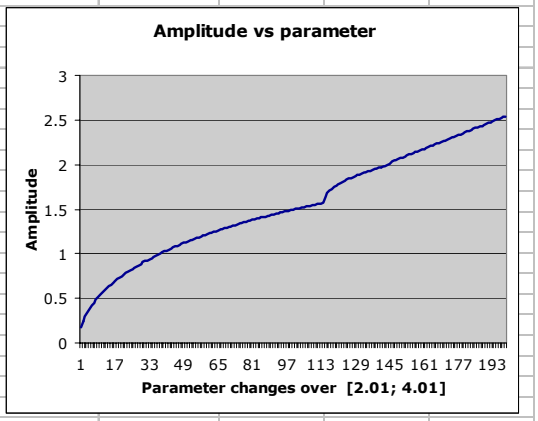



Figure 21: $A(\alpha)$ in equation (6) increases monotonically with the parameter $\alpha$.

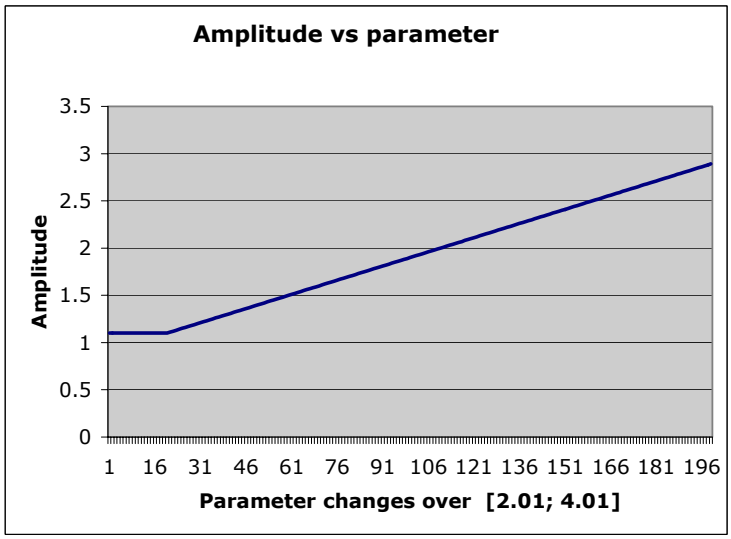



Figure 22: $x(0) = 1.1$

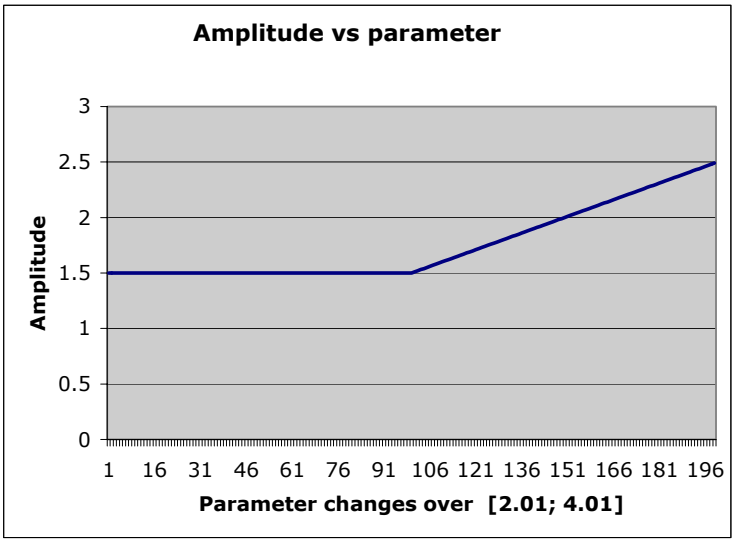



Figure 23: $x(0) = 1.5$

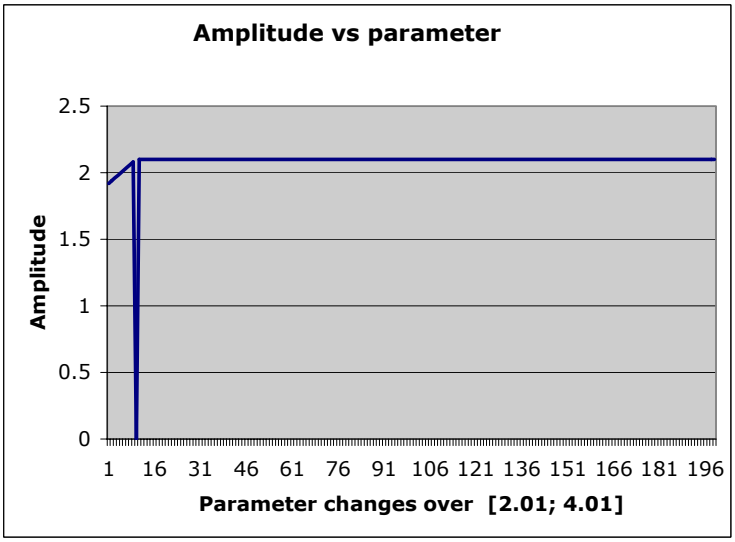



Figure 24: $x(0) = 2.1$

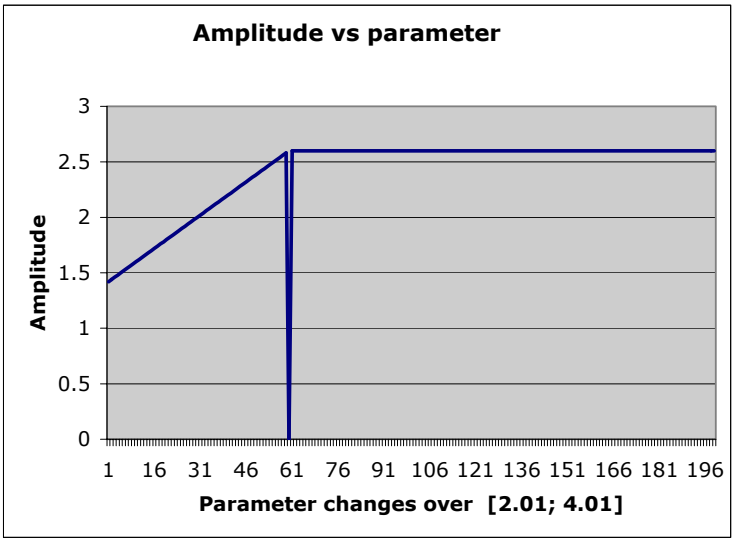



Figure 25: $x(0) = 2.6$

$A(\alpha)$ for different positive values of $x(0)$ suggest two different formulas depending on whether $x(0) < \alpha_{\min}$ or $x(0) > \alpha_{\min}$.

If $0 < x(0) < \alpha_{\min}$, then

$$A(\alpha) = \begin{cases} x(0) & \text{if} \quad \alpha < 2x(0) \\ \alpha - x(0) & \text{if} \quad \alpha > 2x(0) \end{cases}$$

If $x(0) > \alpha_{\min}$, then

$$A(\alpha) = \begin{cases} 2\alpha - x(0) & \text{if} \quad \alpha < x(0) \\ 0 & \text{if} \quad \alpha = x(0) \\ x(0) & \text{if} \quad \alpha > x(0) \end{cases}$$

Similarly, the case of $x(0) < 0$ can be considered.

An interesting fact can be discovered through the computational experiment involving the spreadsheet function `OFFSET` is that there exists $\alpha = \alpha^*$ such that $A(\alpha, x(0)) = x(0)$ for $\alpha > \alpha^*$. In order to locate such a value $\alpha^*$, a means of selecting a value from a designated cell is required. The spreadsheet pictured in Figure 26 shows zero in cell `G3` as the return of the formula `=OFFSET(D6, 100*C6, 1)`, where `C6` relates to the value of parameter $\alpha$ for which the amplitude equals zero. Such value of $\alpha$ can be located through changing the value of $x(0)$ using the slider located above cell `C6`. This can give rise to a mathematical activity aimed at the formal demonstration of the results of a computational experiment. Indeed, as mentioned in [27], the solution to equation (10) with initial data $x(0) \in (0, \alpha)$ can be represented either as $x(t) = x(0)$ or as $x(t) = x(0) - \alpha$, depending on whether $t$ is an even or odd number. This shows that eventually, after the orbits

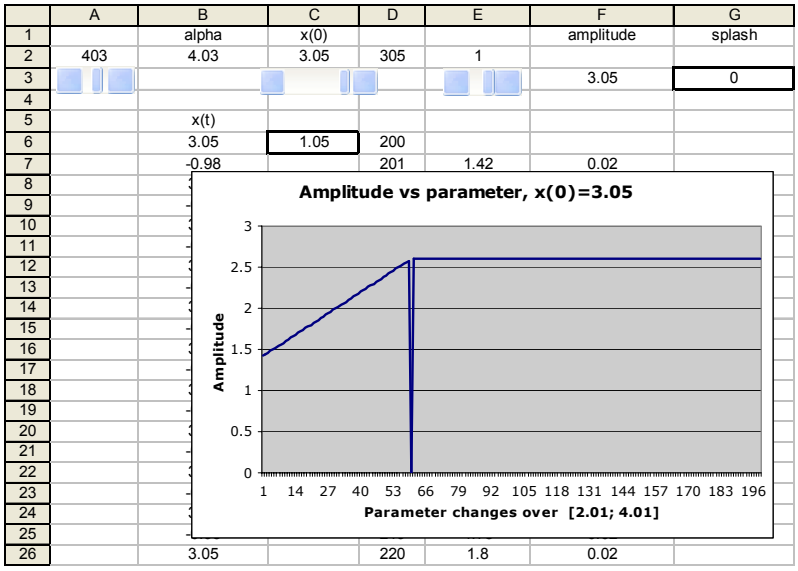



Figure 26: The use of the function OFFSET.

of equation (10) have settled down, they exhibit oscillating behavior with period two, resembling oscillations that occur in two-dimensional conservative continuous dynamic systems [26]. In such a way, through a pedagogy of spreadsheet-based experimentation with digital phase-locked loops, certain phenomena can be observed first, then formally interpreted, and finally, rigorously proved.

## 7 Conclusion

This paper has demonstrated how the combination of a number of computational techniques made possible by the use of a spreadsheet enables educational applications of digital phase-locked loops commonly used in radio engineering, communication, and computer architecture. Described by non-linear difference equations, the behavior of which is extremely complicated from a pure mathematical perspective, these control systems can be found across various tertiary curricula. The use of a spreadsheet allows an alternative approach to the study of this topic – embedding advanced mathematical ideas in the software frees learners from the necessity of dealing with complex mathematical machinery of non-linear control systems that rigorous methods of investigation require. In particular, this alternative approach has been used to introduce this topic to students of discrete mathematics at Bond University.

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