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# View factor for inclined plane with Gaussian source

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# View Factor for Inclined Plane with Gaussian Source

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**Abstract**

The view factor (angle factor) for a differential inclined plane in the case of a radiating source of radially Gaussian intensity is considered. This information is useful for modelling of solar radiation in certain applications. The view factor is expressed in terms of two integrals, one of which is obtained in closed form in terms of special functions, and the other is approximated. A compact estimate for the view factor is presented which is suitable for machine computation. While the relative error associated with the final estimate is typically less than 0.01%, and in all cases, less than 0.2%, the method is easily extended to yield even greater accuracy.

Keywords: solar radiation, view factor, Gaussian source.

## 1 Introduction

View factors for inclined planes for the case of a uniformly radiating source are well-known and can be found in standard texts on radiative transfer, for example [1] or [2]. The purpose of the present work is to derive an accurate estimate of the view factor for an infinite plane source whose intensity falls off radially in accordance with a Gaussian law and a small differential receiving plane surface inclined at some acute angle to the source plane. The final result is intended to be of use for work on modelling the intensity distribution of the apparent solar disc by the Gaussian function; the receiving plane being the Earth's surface. The Gaussian source assumption follows a suggestion of Peck [4] in his work on parabolic solar collector design. The paper is organized as follows. We first consider the geometry of the problem and obtain an exact integral expression for the view factor, as a function of the angle of inclination, $\beta$, of the differential receiving plane. Following this, some standard estimation techniques are employed to derive a useful approximation to the view factor. Finally, this estimate is used as the basis of a computational model in Microsoft Excel, which computes, tabulates, and graphs the view factor as a function of $\beta$.

## 2 Methods of computing view factors

According to Modest[1] [1], view factor evaluation techniques may be classified broadly as follows.

1. Direct integration.
2. Statistical: sampling with the well-known Monte-Carlo class of methods.
3. Special methods.
   (a) Algebraic: application of symmetry, reciprocity, and other properties.
   (b) Crossed-string method: applied to long enclosures with constant cross-section.
   (c) Unit sphere method: used between one infinitesimal and one finite area, such as we have here[2].
   (d) Inside sphere method[3].

Direct integration may be accomplished by any of a number of efficient methods, analytical or numerical, with the common *Simpson's Rule* being very accurate for most shapes, even with modest mesh size. Methods in the *Monte-Carlo* class are typically poorer cousins to the other, preferred methods, and errors are of order $\sqrt{h}$, where $h$ is the mesh size. Such techniques should be chosen only when others are, for whatever reason, very awkward to apply. Industry Standard scientific software such as FACET [3] employs a combination of several of these methods to achieve satisfactory results, in general.

The present method falls into category 1, with the integrals here being evaluated partially analytically, and partially estimated. There is no particular numerical technique used; rather the whole exercise here is primarily one of estimation of semi-tractable integrals in terms of known special functions, and their final estimation using asymptotic techniques. Thus, it makes little sense to compare the present work with some of the more advanced numerical techniques for view factor estimation.

[1] This work appears to be both the most modern and the most complete text dealing with radiative heat transfer, devoting an entire chapter to a discussion of view factors and a comparison of computational methods.

[2] This method is of little value here, given the unusual intensity distribution of the source.

[3] Similar comments as in footnote 2 apply.

## 3 Net incident intensity for arbitrary source distribution function

With reference to Figure 1, the vector $\varphi$ is the position vector to an arbitrary point $P$ in the plane $A_1$. With respect to the right-handed orthonormal system $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$ emanating from $O_2$, we have

$$\varphi = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} \tag{1}$$

The vector $\varphi_0$ is normal to the plane $A_1$:

$$\varphi_0 = \varphi_0 \mathbf{j} \tag{2}$$

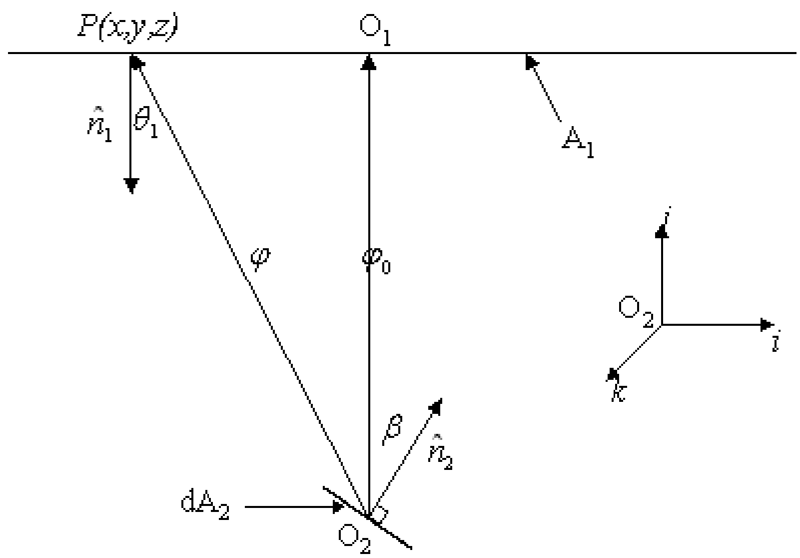


Figure 1: Geometry

Since $P$ lies in $A_1$, we have $y = \varphi_0$, so eq (1) becomes:

$$\varphi = x\mathbf{i} + \varphi_{\mathbf{0}}\mathbf{j} + z\mathbf{k} \tag{3}$$

The angles $\theta_1$ and $\theta_2$ are respectively the angles of emission from $P$ and incidence at $O_2$ of radiation from the source plane $A_1$. The angle $\beta$ is the (acute) angle between the normals, respectively $\mathbf{n_1}$ and $\mathbf{n_2}$, to the planes $A_1$ and $dA_2$.

From figure 1, we have

$$\mathbf{n_2} = (\sin\beta)\,\mathbf{i} + (\cos\beta)\,\mathbf{j} \tag{4}$$

so that

$$\cos\theta_2 = \mathbf{n}_2^T \varphi = \frac{x\sin\beta + \varphi_0\cos\beta}{\varphi} \tag{5}$$

Also,

$$\cos\theta_1 = \frac{\varphi_0}{\varphi} \tag{6}$$

Now the intensity of illumination, $I_2$, on $dA_2$, is given by

$$I_2 = \int\int_{A_1'\subset A_1} \frac{I(P)\cos\theta_1\cos\theta_2}{\varphi^2} dA_1 \tag{7}$$

where $I(P)$ is the intensity of emission at $P$ on $A_1$ and $A_1' = \{P \in A_1 : \cos\theta_2 > 0\}$, i.e., $A_1'$ is the set of all points on $A_1$ giving rise to an acute angle of incidence at $O_2$.

Using eqs (5), (6) in (7), we find that:

$$I_2 = \int\int_{A_1'} \frac{I(P)\left(x\sin\beta + \varphi_0\cos\beta\right)}{\varphi^4} dA_1 \tag{8}$$

Fubini's theorem permits the conversion of this double integral to the following iterated integral:

$$I_2 = \varphi_0 \int_{x_0}^{\infty}\int_{-\infty}^{\infty} \frac{I(P)\left(x\sin\beta + \varphi_0\cos\beta\right)}{\varphi^4} dzdx \tag{9}$$

where $A_1^{'} = [x_0, \infty) \times \Re$. It is a simple matter to determine $x_0$ from problem constants. From eq (5) we deduce that

$$x_0 = -\varphi_0 \cot\beta \tag{10}$$

Hence, using this result and (3), eq (9) becomes:

$$I_2 = \varphi_0 \int_{-\varphi_0\cot\beta}^{\infty}\int_{-\infty}^{\infty} \frac{I(P)\left(x\sin\beta + \varphi_0\cos\beta\right)}{\left(x^2 + \varphi_0^2 + z^2\right)^2} dzdx \tag{11}$$

# 4 Net incident intensity for Gaussian source

In this section we take

$$I(P) = \frac{I_0}{\sigma\sqrt{2\pi}} e^{-\left(x^2+z^2\right)/2\sigma^2} \tag{12}$$

Substituting for $I(P)$ in equation (11), using the transformations $\xi = x/\sigma$ and $\eta = z/\varphi$, and writing $\alpha = \varphi_0^2/2\sigma^2$, we obtain:

$$\frac{\sigma\sqrt{2\pi}I_2}{\varphi_0 I_0} = \frac{\varphi_0\cos\beta}{\sigma^2}\int_{-(\varphi_0\cot\beta)/\sigma}^{\infty} e^{-\xi^2/2}\int_{-\infty}^{\infty}\frac{e^{-\eta^2/2}d\eta d\xi}{\left(2\alpha + \xi^2 + \eta^2\right)^2} \tag{13}$$
$$+ \frac{\sin\beta}{\sigma}\int_{-(\varphi_0\cot\beta)/\sigma}^{\infty} e^{-\xi^2/2}\int_{-\infty}^{\infty}\frac{e^{-\eta^2/2}d\eta d\xi}{\left(2\alpha + \xi^2 + \eta^2\right)^2}$$

Eq (13) may be rewritten as follows:

$$\frac{\sigma}{2}\sqrt{\frac{\pi}{\alpha}}\frac{I_2}{I_0} = \sqrt{2\alpha}\cos\beta\int_{-\sqrt{2\alpha}\cot\beta}^{\infty} e^{-\xi^2/2} J\left(\sqrt{2\alpha+\xi^2}\right)d\xi \qquad (14)$$
$$+\sin\beta\int_{-\sqrt{2\alpha}\cot\beta}^{\infty} \xi e^{-\xi^2/2} J\left(\sqrt{2\alpha+\xi^2}\right)d\xi$$

where,

$$J(t) = \int_0^\infty \frac{e^{-\eta^2/2}d\eta}{\left(t^2+\eta^2\right)^2} \qquad (15)$$

and the evenness of the integrand has been exploited. The function $J$, defined by the integral (15) may be expressed in terms of the function $K$, where

$$K(x) = \int_0^\infty \frac{e^{-xz^2}dz}{\left(1+z^2\right)^2} \qquad (16)$$

The substitution $\eta = tz$ in (15) yields

$$J(t) = \int_0^\infty \frac{e^{-tz^2/2}tdz}{t^4\left(1+z^2\right)^2}$$

so that

$$J(t) = \frac{1}{t^3}K(\frac{t^2}{2}) \qquad (17)$$

It is possible to express $K$ in terms of common functions [5]:

$$K(x) = \frac{\pi}{4}(1-2x)e^x\left(1-\operatorname{erf}\sqrt{x}\right)+\frac{\sqrt{\pi x}}{2} \qquad (18)$$

Combining equations (17) and (18) we find that

$$J(t) = \frac{1}{t^3}\left\{\frac{\pi}{4}(1-t^2)e^{t^2/2}\left(1-\operatorname{erf}\frac{t}{\sqrt{2}}\right)+\sqrt{\frac{\pi}{2}}\frac{t}{2}\right\} \qquad (19)$$

Writing

$$S(\alpha,\beta) = \int_{-\sqrt{2\alpha}\cot\beta}^{\infty} \xi e^{-\xi^2/2} J\left(\sqrt{2\alpha+\xi^2}\right)d\xi \qquad (20)$$

and

$$R(\alpha,\beta) = \int_{-\sqrt{2\alpha}\cot\beta}^{\infty} e^{-\xi^2/2} J\left(\sqrt{2\alpha+\xi^2}\right)d\xi \qquad (21)$$

we may express eq (14) more concisely as:

$$\frac{\sigma}{2}\sqrt{\frac{\pi}{\alpha}}\frac{I_2}{I_o} = \sqrt{2\alpha}\cos\beta R(\alpha,\beta)+\sin\beta S(\alpha,\beta) \qquad (22)$$

Turning our attention now to the integral $S(\alpha, \beta)$, as defined by (20), it will be found that substitution for $J$ from eq (19) in the integrand of eq (20) yields the following expression for $S(\alpha, \beta)$:

$$
\begin{aligned}
S(\alpha, \beta) = \frac{\pi e^{\alpha}}{4} \int\limits_{-\sqrt{2\alpha}\cot\beta}^{\infty} \frac{\xi\left(1 - 2\alpha - \xi^2\right)\left(1 - \operatorname{erf}\sqrt{\alpha + \xi^2/2}\right)}{\left(2\alpha + \xi^2\right)^{3/2}} d\xi \\
+ \frac{\sqrt{\pi}}{2} \int\limits_{-\sqrt{2\alpha}\cot\beta}^{\infty} \frac{\xi e^{-\xi^2/2}}{\sqrt{2\alpha + \xi^2}} d\xi
\end{aligned}
\qquad (23)
$$

Appropriate changes of variable reduce eq (23) to a somewhat more tractable form. Put $\gamma = \sqrt{\alpha}/\sin\beta$ and in the first integral, take $u = \sqrt{\alpha + \xi^2/2}$, while in the second, $u = \alpha + \xi^2/2$, yielding:

$$
S(\alpha, \beta) = \frac{\sqrt{\pi} e^{\alpha}}{2\sqrt{2}} \left[\sqrt{\pi}\left(\gamma + \frac{1}{2\gamma}\right)(1 - \operatorname{erf}\gamma) - e^{-\gamma^2}\right] \qquad (24)
$$

For the remaining integral in eq (22), $R(\alpha, \beta)$, the substitution $\xi = -\sqrt{2\alpha}\cot\theta$ reduces eq (21) to the following angular integral:

$$
R(\alpha, \beta) = \int_{\beta}^{\pi} e^{-\alpha\cot^2\beta} J(\sqrt{2\alpha}/\sin\theta) d\theta
$$

and using eq (17), we obtain:

$$
R(\alpha, \beta) = \frac{1}{2\alpha} \int_{\beta}^{\pi} \sin\theta e^{-\alpha\cot^2\beta} K(\alpha/\sin^2\theta) d\theta \qquad (25)
$$

Eq (18) may now be used to write the integrand of (25) in terms of elementary functions:

$$
\begin{aligned}
4\alpha R(\alpha, \beta) = \frac{\pi e^{\alpha}}{2} \int_{\beta}^{\pi} (\sin\theta - 2\alpha/\sin\theta) \operatorname{erf} c\left(\sqrt{\alpha}/\sin\theta\right) d\theta \\
+ \sqrt{\pi\alpha} \int_{\beta}^{\pi} e^{-\alpha\cot^2\theta} d\theta
\end{aligned}
\qquad (26)
$$

In order to establish an asymptotic representation for $R(\alpha, \beta)$,valid as $\alpha \to +\infty$ through real values, and with $\beta$ as a parameter, the following well-known asymptotic expansion for the complementary error function will be employed; see, for example, Abramowitz & Stegun, [6, sec 7.1.23, 7.1.24].

$$
1 - \operatorname{erf} x = \frac{e^{-x^2}}{x\sqrt{\pi}} \left(1 - \frac{1}{2x^2} + r(x)\right) \qquad (27)
$$

$$
(x \to +\infty)
$$

where

$$|r(x)| < \frac{3}{4x^4} \tag{28}$$

Using eq (27) in eq (26), we obtain:

$$\begin{aligned} 4\sqrt{\frac{\alpha}{\pi}} R(\alpha,\beta) &= \frac{1}{2\alpha}\int_\beta^\pi e^{-\alpha\cot^2\theta}\sin^2\theta\left(2-\frac{1}{2\alpha}\sin^2\theta\right)d\theta \\ &+ \frac{1}{2\alpha}\int_\beta^\pi e^{-\alpha\cot^2\theta}\left(\sin^2\theta-2\alpha\right) r(\sqrt{\alpha}/\sin\theta)d\theta \\ &\equiv P(\alpha,\beta)+r^*(\alpha,\beta) \end{aligned} \tag{29}$$

In eq (29), the first integral, $P(\alpha,\beta)$, is our estimate and the second, $r^*(\alpha,\beta)$, is the remainder. $P(\alpha,\beta)$ may be split into two integrals, each of which may be handled by the method of Laplace [7, p81] to yield asymptotic representations. We have:

$$P(\alpha,\beta) = \frac{1}{\alpha}\int_\beta^\pi q_1(\theta)e^{-p(\theta)}d\theta - \frac{1}{4\alpha^2}\int_\beta^\pi q_2(\theta)e^{-p(\theta)}d\theta$$

in which $q_1(\theta)=\sin^2\theta, q_2(\theta)=\sin^4\theta, p(\theta)=\cot^2\theta$.

The minimum of $p(\theta)$ occurs at $\theta=\pi/2$, an interior point of the interval of integration. Accordingly, expand the functions $p(\theta), q_1(\theta), q_2(\theta)$ about $\theta=\pi/2$ and retain only dominant terms, to yield:

$$\begin{aligned} P(\alpha,\beta) &= \left(\frac{1}{\alpha}\sqrt{\frac{\pi}{\alpha}}+o(\alpha^{-3/2})\right)-\left(\frac{1}{4\alpha^2}\sqrt{\frac{\pi}{\alpha}}+o(\alpha^{-5/2})\right) \\ &= \frac{1}{\alpha}\sqrt{\frac{\pi}{\alpha}}+o(\alpha^{-3/2}) \qquad (\alpha\to+\infty) \end{aligned} \tag{30}$$

As for the remainder, $r^*(\alpha,\beta)$, we have:

$$\begin{aligned} |r^*(\alpha,\beta)| &= \left|\int_\beta^\pi e^{-\alpha\cot^2\theta}\left(\frac{\sin^2\theta}{2\alpha}-1\right) r(\sqrt{\alpha}/\sin\theta)d\theta\right| \\ &< \int_\beta^\pi e^{-\alpha\cot^2\theta}\left|r(\sqrt{\alpha}/\sin\theta)\right|d\theta \qquad \text{for } \alpha>\frac{1}{2} \\ &< \frac{3}{4\alpha^2}\int_\beta^\pi \sin^4\theta e^{-\alpha\cot^2\theta}d\theta \qquad \text{using eq (28)} \\ &< \frac{3}{4\alpha^2}\int_0^\pi \sin^4\theta e^{-\alpha\cot^2\theta}d\theta \end{aligned}$$

The substitution $u=-\cot\theta$ leads to:

$$|r^*(\alpha,\beta)| < \frac{3}{2\alpha^2}\int_0^\infty \left(1+u^2\right)^{-3}e^{-\alpha u^2}du \tag{31}$$

A crude estimate for this last integral is adequate for the present purpose.

$$|r^*(\alpha,\beta)| < \frac{3}{2\alpha^2}\int_0^\infty e^{-\alpha u^2} du = \frac{3}{2\alpha^2}\sqrt{\frac{\pi}{\alpha}}$$

We conclude that

$$|r^*(\alpha,\beta)| = o(\alpha^{-2}) \tag{32}$$

Combining (29), (30) and (32), we have:

$$R(\alpha,\beta) = \frac{\pi}{4\alpha^2} + o(\alpha^{-2}) \tag{33}$$

# 5 An estimate for $R(\alpha,\beta)$ with error bounds

The asymptotic estimate of eq (33) for $R(\alpha,\beta)$ may be useful for some applications, but does not provide error bounds suitable for the construction of a reasonably accurate tabulation of $R(\alpha,\beta)$ versus $\beta$, with $\alpha$ fixed. Since the primary aim of the present work is to derive computationally useful approximations to the net view factor, we now consider alternative methods. One possibility is integration by parts, which certainly gives an explicit expression for the error term; another is the use of Watson's Lemma [7, p71], which yields a useful approximation plus an acceptable bound for the error. Here, we choose integration by parts, because of its simplicity and ability to obtain approximations for a large range of the parameter of interest, with quite a small relative error.

Proceeding with the analysis, we have from eq (29):

$$4\sqrt{\frac{\alpha}{\pi}}R(\alpha,\beta) = Q(\alpha,\beta) + r^+(\alpha,\beta) \tag{34}$$

where

$$Q(\alpha,\beta) = \frac{1}{\alpha}\int_\beta^\pi e^{-\alpha\cot^2\theta}\sin^2\theta d\theta \tag{35}$$

and

$$r^+(\alpha,\beta) = \int_\beta^\pi e^{-\alpha\cot^2\theta}\left(\frac{\sin^2\theta}{2\alpha} - 1\right) r(\sqrt{\alpha}/\sin\theta)d\theta + \frac{1}{4\alpha^2}\int_\beta^\pi e^{-\alpha\cot^2\theta}\sin^4\theta d\theta \tag{36}$$

Now from (36), we obtain:

$$\left|r^+(\alpha,\beta)\right| < \int_\beta^\pi e^{-\alpha\cot^2\theta}\left|r(\sqrt{\alpha}/\sin\theta)\right| d\theta + \frac{1}{4\alpha^2}\int_\beta^\pi e^{-\alpha\cot^2\theta}\sin^4\theta d\theta; \qquad \text{if } \alpha > 1/2$$

Using (28), we have, for $\alpha > 1/2$:

$$\begin{aligned}\left|r^+(\alpha,\beta)\right| &< \int_\beta^\pi \frac{3\sin^4\theta}{4\alpha^2} e^{-\alpha\cot^2\theta} d\theta + \frac{1}{4\alpha^2}\int_\beta^\pi e^{-\alpha\cot^2\theta}\sin^4\theta d\theta \\ &= \frac{1}{\alpha^2}\int_\beta^\pi e^{-\alpha\cot^2\theta}\sin^4\theta d\theta \\ &< \frac{1}{\alpha^2}\int_\beta^\pi e^{-\alpha\cot^2\theta}\sin^2\theta d\theta\end{aligned}$$

Comparing this last inequality with eq (35), we have:

$$\left|r^+(\alpha,\beta)\right| < \frac{1}{\alpha} Q(\alpha,\beta) \tag{37}$$

We have, therefore,

$$4\sqrt{\frac{\alpha}{\pi}} R(\alpha,\beta) = Q(\alpha,\beta)\left(1+\tau(\alpha,\beta)\right) \tag{38}$$

where

$$\tau(\alpha,\beta) = \frac{|r^+(\alpha,\beta)|}{Q(\alpha,\beta)} \tag{39}$$

and

$$0 < \tau(\alpha,\beta) < \frac{1}{\alpha} \tag{40}$$

It is convenient to make the substitution $u = -\cot\theta$ in eq (35), so that

$$\alpha Q(\alpha,\beta) = \int_{-\cot\beta}^{\infty} e^{-\alpha u^2}(1+u^2)^{-2} du$$

or,

$$\alpha Q(\alpha,\beta) = \int_0^{\cot\beta} e^{-\alpha u^2}(1+u^2)^{-2} du + \int_0^{\infty} e^{-\alpha u^2}(1+u^2)^{-2} du \tag{41}$$

Using notation defined by eqs (16) and (74), we may write eq (41) as:

$$\alpha Q(\alpha,\beta) = Y(\alpha,\cot\beta) + K(\alpha) \tag{42}$$

Appendices A and B provides estimates with error bounds for $Y(\alpha, x)$ for $x \geq 1$ and $0 \leq x \leq 1$ respectively, while Appendix C has an asymptotic representation with error bound for $K(\alpha)$. In order to make use of these results, in eq(42), it is necessary to dissect the interval in which $\cot\beta$ lies, viz., $[0,\infty)$ into the two subintervals $[0,1]$ and $(1,\infty)$. Thus, we consider the following two cases.

1. $0 \leq \cot\beta \leq 1$, i.e., $\pi/4 \leq \beta \leq \pi/2$, and
2. $\cot\beta > 1$, i.e., $0 < \beta < \pi/4$.

Note that if $\beta = 0$, eq (42) implies that $\alpha Q(\alpha, 0) = 2K(\alpha)$, which may be conveniently included in case 2 above by allowing $\beta \to 0$. The details will emerge later.

### 5.1 Case 1: $0 \le \cot\beta \le 1$

From eqs (42), (109) and (114), we have:

$$\alpha Q(\alpha,\beta) = \frac{\sqrt{\pi}}{2\sqrt{\alpha}}\left(1 + \operatorname{erf}\left(\sqrt{\alpha}\cot\beta\right) + \eta^*(\alpha) - \rho\operatorname{erf}\left(\sqrt{\alpha}\cot\beta\right)\right) \tag{43}$$

We define the error term

$$\rho^*(\alpha,\beta) \equiv \eta^*(\alpha) - \rho\operatorname{erf}\left(\sqrt{\alpha}\cot\beta\right) \tag{44}$$

and bounds on $\eta^*(\alpha)$ and $\rho$ are given by (116) and (110) respectively. These may be used to obtain a useful bound for $\rho^*(\alpha,\beta)$.

$$\begin{aligned}
|\rho^*(\alpha,\beta)| &< |\eta^*(\alpha)| + \rho\operatorname{erf}\left(\sqrt{\alpha}\cot\beta\right)\\
&< |\eta^*(\alpha)| + \rho\\
&< \frac{1}{\alpha} + \frac{1}{\alpha-1}\\
&< \frac{2}{\alpha-1}
\end{aligned}$$

In fact, $0 < -\rho^*(\alpha,\beta) < 2/(\alpha-1)$ since $\rho^* < 0$. We may now write eq (43) in the form:

$$\alpha Q(\alpha,\beta) = \frac{\sqrt{\pi}}{2\sqrt{\alpha}}\left(1 + \operatorname{erf}\left(\sqrt{\alpha}\cot\beta\right) + \rho^*\right) \tag{45}$$

where

$$0 < -\rho^*(\alpha,\beta) < \frac{2}{(\alpha-1)} \tag{46}$$

Returning to eq (102) and using (109), we have:

$$R(\alpha,\beta) = \frac{\pi}{8\alpha^2}\left(1 + \operatorname{erf}\left(\sqrt{\alpha}\cot\beta\right) + \rho^*\right)(1 + \tau(\alpha,\beta)) \tag{47}$$

Define our error term $\tau^*$:

$$\tau^* = \tau\left[1 + \operatorname{erf}\left(\sqrt{\alpha}\cot\beta\right)\right] + \rho^*(1 + \tau(\alpha,\beta))$$

We may now write

$$\begin{aligned}
R(\alpha,\beta) &= \frac{\pi}{8\alpha^2}\left(1 + \operatorname{erf}\left(\sqrt{\alpha}\cot\beta\right) + \tau^*\right)\\
0 &\le \cot\beta \le 1
\end{aligned} \tag{48}$$

Using eqs 46 and 40, and the fact that erf $< 1$, we may obtain a bound for the error term $\tau^*$as follows.

$$\begin{aligned}
|\tau^*| &< 2\tau + \rho^*(1+\tau)\\
&< 2\tau + \frac{2}{\alpha-1}(1+\tau)\\
&< \frac{2}{\alpha} + \frac{2}{\alpha-1}\left(1 + \frac{1}{\alpha}\right)
\end{aligned}$$

Thus, we have:

$$|\tau^*| < \frac{4}{\alpha - 1} \tag{49}$$

### 5.2 Case 2: $\cot\beta > 1$

From eq (41), we have

$$\alpha Q(\alpha,\beta) = 2K(\alpha) - \int_{\cot\beta}^{\infty} e^{-\alpha u^2}(1+u^2)^{-2}du \tag{50}$$

Define

$$\widetilde{\eta}(\alpha,\beta) = \int_{\cot\beta}^{\infty} e^{-\alpha u^2}(1+u^2)^{-2}du \tag{51}$$

We now consider $2K(\alpha)$ as the approximation to $\alpha Q(\alpha,\beta)$ and $\widetilde{\eta}(\alpha,\beta)$ as the error, and seek a bound for $\widetilde{\eta}(\alpha,\beta)$. Since $2K(\alpha) \sim \sqrt{\pi/\alpha}$, a bound of lesser order is sought. Fortunately, such a bound is easy to obtain. Eq (51) yields

$$\begin{aligned}\widetilde{\eta}(\alpha,\beta) &< e^{-\alpha\cot^2\beta}\int_{\cot\beta}^{\infty}(1+u^2)^{-2}du \\ &< e^{-\alpha}\int_{0}^{\infty}(1+u^2)^{-2}du\end{aligned}$$

Thus, we have

$$\widetilde{\eta}(\alpha,\beta) < \frac{\pi}{4}e^{-\alpha} \tag{52}$$

Since $\alpha$ is large, it is clear that $\alpha Q(\alpha,\beta)$ is approximated very well by $2K(\alpha)$, whenever $\cot\beta > 1$. In the physical situation, i.e., Sun-Earth radiation path, $\alpha > 20{,}000$, and so there seems little point in working through a complete error analysis. It is more expedient to simply ignore the exponentially small term $e^{-\alpha}$ in comparison to other terms like $1/\alpha$. So, for $\cot\beta > 1$, we commit an exceedingly small error by taking $\alpha Q(\alpha,\beta) \approx 2K(\alpha)$. For $\cot\beta > 1$ and $-1/\alpha < \eta^*(\alpha) < 0$, this leads to:

$$R(\alpha,\beta) = \frac{\pi}{4\alpha^2}\left(1+\eta^*(\alpha)\right) \tag{53}$$

Indeed, eq (48) is valid (within the given bound for $\tau^*$) for *unrestricted* positive $\cot\beta$, since if $\cot\beta > 1$,then $\operatorname{erf}(\sqrt{\alpha}\cot\beta)$ differs from unity by less than the following quantity [6, sec 7.1.23, 7.1.24].

$$\frac{1}{\sqrt{\pi\alpha}\cot\beta}e^{-\alpha\cot^2\beta} < e^{-\alpha}$$

From eq (48), it is clear that the uniform $\beta \in [0,\pi/2]$ approximation $1 + \operatorname{erf}(\sqrt{\alpha}\cot\beta)$ to the expression $8\alpha^2 R(\alpha,\beta)/\pi$ incurs a relative error of less than $|\tau^*|$, since $0 \le \operatorname{erf} \le 1$. Finally then, we have the result:

$$R(\alpha,\beta) \simeq \frac{\pi}{8\alpha^2}\left(1+\operatorname{erf}\left(\sqrt{\alpha}\cot\beta\right)\right) \tag{54}$$

This last estimate has a relative error of less than $4/(\alpha-1)$.

# 6 An estimate for $S(\alpha, \beta)$ with error bounds

Eq (93) gives a closed form expression for $S(\alpha, \beta)$ in terms of commonly tabulated functions. Since this expression makes reference to the error function $\mathrm{erf}(x)$, in order to obtain an estimate for $S(\alpha, \beta)$ in terms of exponential and rational functions of $\gamma$, where $\gamma = \sqrt{\alpha}/\sin\beta$, we now need some estimate of this function. It is found that, if the expression of eq (24) is naïvely evaluated; for example, by coding into some programming language with a library call to erf, significant loss of accuracy results because of cancellation of high order terms. It is preferred therefore to use an appropriate asymptotic expansion for erf so that we may observe this cancellation algebraically, and hence determine how many terms we need for the final estimate. The asymptotic expansion of Abramowitz & Stegun, [6, sec 7.1.23] is used, taking three terms plus remainder. The two terms of highest order cancel with other terms in the expression (93) for $S(\alpha, \beta)$. The relevant approximation to erf $(\gamma)$ is given by:

$$\sqrt{\pi}\left(1 - \mathrm{erf}\left(\gamma\right)\right) = \frac{e^{-\gamma^2}}{\gamma}\left(1 - \frac{1}{2\gamma^2} + \frac{3}{4\gamma^4} - R_3\left(\gamma\right)\right) \tag{55}$$

where

$$0 < R_3\left(\gamma\right) < \frac{15}{8\gamma^6} \tag{56}$$

Substitution of eq (55) in eq (93) for $\sqrt{\pi}\left(1 - \mathrm{erf}\left(\gamma\right)\right)$ gives:

$$S(\alpha, \beta) = \frac{\sqrt{\pi}e^{\alpha-\gamma^2}}{4\sqrt{2}\gamma^4}\left[1 + \epsilon\left(\gamma\right)\right] \tag{57}$$

where

$$\epsilon\left(\gamma\right) = \frac{3}{16\gamma^2} - \gamma^2\left(1 + 2\gamma^2\right)R_3\left(\gamma\right) \tag{58}$$

Using the triangle inequality, and (56), we have:

$$\left|\epsilon\left(\gamma\right)\right| < \frac{63}{16\gamma^2} + \frac{15}{8\gamma^4} \tag{59}$$

Finally, since $\gamma = \sqrt{\alpha}/\sin\beta$, eq (57) becomes:

$$S(\alpha, \beta) = \frac{\sqrt{\pi}e^{-\alpha\cot\beta}\sin^4\beta}{4\sqrt{2}\alpha^2}\left[1 + \epsilon\right] \tag{60}$$

with

$$|\epsilon| < \frac{63\sin^2\beta}{16\alpha} + \frac{15\sin^4\beta}{8\alpha^2} \tag{61}$$

# 7 Final approximation for the view factor $I_2/I_0$

We are now in a position to use the approximate representation for $S(\alpha,\beta)$ and $R(\alpha,\beta)$ (eqs (60) and (54) respectively) to construct a fairly accurate approximation to the net view factor $I_2/I_0$. The expression for $I_2/I_0$ in terms of $S(\alpha,\beta)$ and $R(\alpha,\beta)$ is given by eq (22). It is

$$\frac{\sigma}{2}\sqrt{\frac{\pi}{\alpha}}\frac{I_2}{I_0} = \sqrt{2\alpha}\cos\beta R(\alpha,\beta) + \sin\beta S(\alpha,\beta) \tag{62}$$

Using the expressions for $S(\alpha,\beta)$ and $R(\alpha,\beta)$ from eqs (60) and (54), eq (62) becomes:

$$\begin{aligned}\frac{\sigma}{2}\sqrt{\frac{\pi}{\alpha}}\frac{I_2}{I_0} &= \sqrt{2\alpha}\cos\beta\frac{\pi}{8\alpha^2}\left(1+\operatorname{erf}\left(\sqrt{\alpha}\cot\beta\right)+\tau^*(\alpha,\beta)\right)\\ &\quad+\sin\beta\frac{\sqrt{\pi}}{4\sqrt{2}}\frac{\sin^4\beta}{\alpha^2}e^{-\alpha\cot^2\beta}\left(1+\epsilon(\alpha,\beta)\right)\end{aligned} \tag{63}$$

in which

$$|\tau^*(\alpha,\beta)| < \frac{4}{\alpha-1} \tag{64}$$

$$|\epsilon(\alpha,\beta)| < \frac{63}{16\gamma^2}+\frac{15}{8\gamma^4} \tag{65}$$

$$\gamma = \sqrt{\alpha}/\sin\beta \tag{66}$$

Eq (63) may be simplified somewhat:

$$\frac{4\sigma\alpha I_2}{\sqrt{2}I_0} = \sqrt{\pi}\cos\beta\left(1+\operatorname{erf}\left(\sqrt{\alpha}\cot\beta\right)\right)+\frac{1}{\sqrt{\alpha}}\sin^5\beta e^{-\alpha\cot^2\beta}+\varsigma(\alpha,\beta) \tag{67}$$

In this last equation, we have made use of the total error term, $\varsigma(\alpha,\beta)$, where:

$$\varsigma(\alpha,\beta) = \sqrt{\pi}\tau^*(\alpha,\beta)+\frac{1}{\sqrt{\alpha}}\sin^5\beta e^{-\alpha\cot^2\beta}\epsilon(\alpha,\beta) \tag{68}$$

From eqs (64) and (65) the following upper bound for $|\varsigma(\alpha,\beta)|$ may be derived:

$$|\varsigma(\alpha,\beta)| < \frac{4\sqrt{\pi}}{\alpha-1}+\frac{1}{\sqrt{\alpha}}\sin^5\beta e^{-\alpha\cot^2\beta}\left(\frac{63}{16\gamma^2}+\frac{15}{8\gamma^4}\right)$$

and since $\gamma = \sqrt{\alpha}/\sin\beta$, we have:

$$\begin{aligned}|\varsigma(\alpha,\beta)| &< \frac{4\sqrt{\pi}}{\alpha-1}+\frac{1}{\sqrt{\alpha}}\sin^5\beta e^{-\alpha\cot^2\beta}\left(\frac{63\sin^2\beta}{16\alpha}+\frac{15\sin^4\beta}{8\alpha^2}\right)\\ &< \frac{4\sqrt{\pi}}{\alpha-1}+\frac{1}{\sqrt{\alpha}}\sin^5\beta e^{-\alpha\cot^2\beta}\left(\frac{63\sin^2\beta}{16\alpha}+\frac{15\sin^2\beta}{8\alpha}\right)\\ &= \frac{4\sqrt{\pi}}{\alpha-1}+\frac{93}{16}\alpha^{-3/2}\sin^7\beta e^{-\alpha\cot^2\beta}\end{aligned} \tag{69}$$

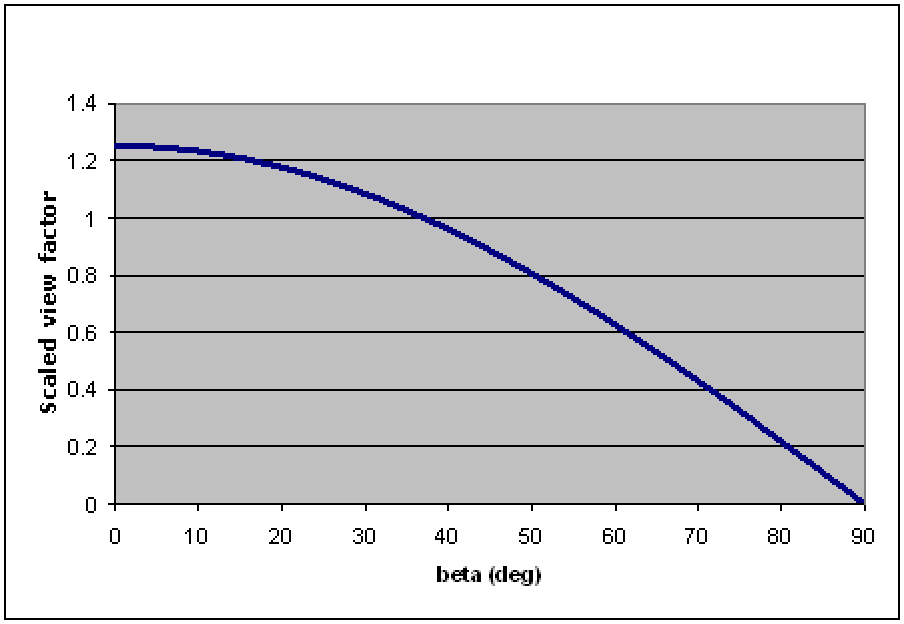


Figure 2: View Factor versus angle of incidence, $\beta$.

The approximation (67) for $4\sigma\alpha I_2/\sqrt{2}I_0$ is associated with an absolute error $|\varsigma(\alpha,\beta)|$, a bound for which is given by (69). From the next section, it is seen that, for almost all values of $\beta$, the relative error of this approximation is less than $10^{-4}$.

A Microsoft Excel model was constructed to evaluate and graph $I_2/I_0$ for specific $\alpha = 22961$ and $0 \leq \beta \leq 90^0$ with increments of $1^0$ using the estimate of eq (67). The graph appears in Figure 2.

# 8 The special case of parallel source and absorber planes ($\beta = 0$)

The view factor for the case $\beta = 0$ has been determined by Peck [4, App C, eq C.13]. Denoting the exponential integral function by $E_1()$, Peck's estimate is given by:

$$\left.\frac{I_2}{I_0}\right|_{\beta=0} = \frac{1}{\sigma}\sqrt{\frac{\pi}{2}}\left(1 - E_1(\alpha)\right) \tag{70}$$

He approximates this expression by:

$$\left.\frac{I_2}{I_0}\right|_{\beta=0} \simeq \sqrt{2\pi}\frac{\sigma}{\varphi_0^2} = \sqrt{\frac{\pi}{2}}\frac{1}{\alpha\sigma} \tag{71}$$

From eq (22), we have, using (54), and finally, ignoring $\tau^*$:

$$\frac{\sigma}{2}\sqrt{\frac{\pi}{\alpha}}\left.\frac{I_2}{I_0}\right|_{\beta=0} = \sqrt{2\alpha}R\left(\alpha,0\right) = \sqrt{2\alpha}\frac{\pi}{8\alpha^2}\left(1+1+\tau^*\right) \simeq \sqrt{2\alpha}\frac{\pi}{4\alpha^2}$$

From this we arrive at:

$$\left.\frac{I_2}{I_0}\right|_{\beta=0} \simeq \frac{2}{\sigma}\frac{\sqrt{\alpha}}{\sqrt{\pi}}\sqrt{2\alpha}\frac{\pi}{4\alpha^2} = \sqrt{\frac{\pi}{2}}\frac{1}{\alpha\sigma}$$

This is in agreement with Peck's estimate, given by the present eq (71).

# 9 Discussion and conclusion

We have presented a technique for the estimation of an unusual view factor in which the source is of Gaussian intensity, as suggested in [4]. The receiving body is a differential plane surface element. The necessary integrals have been partially evaluated exactly and partially estimated, with the final result computed very rapidly in terms of known functions and accurate in most cases to 0.01%. The final estimates are simple enough to quickly enter into a spreadsheet for tabulation and graphing of the view factor as a function of angle of incidence.

The whole exercise is done for a fixed geometry and orientation, and moreover, is not intended to be immediately applied to the case of moving bodies. Recomputation after designated time intervals is therefore not required and a one-off estimation and calculation are all that is needed. This computation is rapid and may be easily implemented in a common programming language such as C, Java, Pascal, or as we have done, in Microsoft's Excel spreadsheet. Further, the complexity inherent in the present analysis stems largely from the assumed intensity distribution of the source, rather than any geometric features. Thus, we have a somewhat unusual approach to a very specific and highly regular problem.

# 10 Acknowledgment

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# References

[1] Modest, M.F. (2003). *Radiative Heat Transfer, 2nd edition.* Academic Press, California. ISBN 0-12-503163-7.

[2] Sparrow, E.M. and Cess, R.D. (1966). *Radiation Heat Transfer*, Brooks-Cole. ISBN 0-89116-923-7.

[3] ———, Features of FACET view factor software are described at URL: www.osti.gov/estsc/PDFs/facet.pdf.

[4] Peck, M.K. (1976). A Ray-Trace Technique for Solar Concentrators, MEngSc Thesis, Department of Mechanical Engineering, University of Queensland.

[5] Sugden, S.J. (2002). *Estimating a Planar View Factor for Source Radiation of Radially Gaussian Intensity*. Bond University School of Information Technology Technical Report #TR02-03.

[6] Abramowitz , M. and Stegun, I.A. (1972). *Handbook of Mathematical Functions*, Dover, 10th printing, with correction, ISBN 0-486-61272-4.

[7] Olver, F.W.J. (1974). *Asymptotics & Special Functions*. Academic Press, ISBN 0-12-525850-X.

# A Estimation of $\Psi(\alpha, x)$

We estimate the integral

$$\Psi(\alpha, x) = \int_x^\infty \frac{e^{-\alpha(u^2-x^2)}du}{(1+u^2)^2} \tag{72}$$

$$\text{for } x \geq (2\alpha - 5)^{-1/4} = \epsilon \tag{73}$$

The method of integration by parts is used to obtain an approximation for this integral, where $\alpha$ is regarded as a large positive real constant. Two successive integrations by parts yield the required approximation, with a relative error of less than $\epsilon^4(3 + 14\epsilon^2 + 35\epsilon^4)$ when $x \geq \epsilon$, and less than $13/\left(\alpha^2 x^2 \left(1 + x^2\right)\right)$ when $x \geq 1$. Proceeding with the method, we first define

$$Y(\alpha, x) = \int_0^x e^{-\alpha u^2} \left(1 + u^2\right)^{-2} du \tag{74}$$

$$Y_c(\alpha, x) = \int_x^\infty e^{-\alpha u^2} \left(1 + u^2\right)^{-2} du \tag{75}$$

We have:

$$Y(\alpha, x) + Y_c(\alpha, x) = K(\alpha) \tag{76}$$

where the function $K$ is defined by eq (16).

Now make the substitution $t = u^2 - x^2$ in equation (72), yielding:

$$\Psi(\alpha, x) = \int_x^\infty e^{-\alpha t} q(t) dt \tag{77}$$

where

$$q(t) = \frac{\sqrt{t + x^2}}{2\left(1 + t + x^2\right)^2} \tag{78}$$

and $x$ is to be regarded as a fixed, but otherwise arbitrary positive parameter.

The expression in eq (77) for $\Psi(\alpha, x)$ is a Laplace integral for each fixed $x$. Standard techniques such as integration by parts or application of Watson's Lemma [7, p71] may be used to obtain an asymptotic expansion for the integral, as $\alpha \to \infty$, provided that the function $q(t)$ satisfies certain conditions (see, for example, [7, pp67-72]. The method used here is that of integration by parts (twice). We have:

$$\Psi(\alpha, x) = \frac{q(0)}{\alpha} + \frac{q'(0)}{\alpha^2} + \frac{1}{\alpha^2}\int_0^\infty e^{-\alpha t} q''(t)dt \tag{79}$$

Now write $\xi = t + x^2$. Since $x$ is constant, we have $d/dt = d/d\xi$. We have:

$$2q(t) = (1+\xi)^{-2}\,\xi^{-1/2} \tag{80}$$

and hence

$$q'(t) = -\frac{1}{4}\left(5+\xi^{-1}\right)\xi^{-1/2}\left(1+\xi\right)^{-3} \tag{81}$$

Also,

$$q''(t) = \frac{1}{8}\xi^{-5/2}\left(1+\xi\right)^{-4}\left(35\xi^2+14\xi+3\right) \tag{82}$$

Since $q''(t)$ is a positive, decreasing function of $t$, and for $x > 0$, its maximum occurs at $t = 0$, i.e., at $\xi = x^2$. That is, we have, for $x > 0$:

$$0 < q''(t) \leq \frac{1}{8}x^{-5}\left(1+x^2\right)^{-4}\left(35x^4+14x^2+3\right) \tag{83}$$

From eqs (80) and (81), we derive:

$$q(0) = x^{-1}\left(1+x^2\right)^{-2} \tag{84}$$

$$q'(0) = -\frac{1}{4}\left(5+\frac{1}{x^2}\right)\frac{1}{x}\left(1+x^2\right)^3 \tag{85}$$

We may therefore write eq (79) as:

$$\Psi(\alpha, x) = A(\alpha, x) + r(\alpha, x) \tag{86}$$

where the approximation is:

$$A(\alpha, x) = \frac{1}{\alpha}\left(\frac{1}{x\left(1+x^2\right)^2}\right) - \frac{1}{\alpha^2}\left(\frac{5+1/x^2}{4x\left(1+x^2\right)^3}\right) \tag{87}$$

and the error (remainder) is:

$$r(\alpha, x) = \frac{1}{\alpha^2}\int_0^\infty e^{-\alpha t} q''(t)dt \tag{88}$$

Using eq 83, a useful upper bound may be obtained for $r(\alpha, x)$ as follows.

$$\begin{aligned} r(\alpha, x) &< \frac{1}{\alpha^2} \max_{[0,\infty)} q''(t) \int_0^\infty e^{-\alpha t} dt \\ &= \frac{35x^4 + 14x^2 + 3}{8\alpha^3 x^5 \left(1 + x^2\right)^4} \end{aligned} \tag{89}$$

It is clear from eqs (88) and (82) that $r(\alpha, x) > 0$, but perhaps not immediately obvious that $A(\alpha, x)$ is also positive (for sufficiently large $\alpha$). Indeed, the estimate $A(\alpha, x)$ for the integral $\Psi(\alpha, x)$ would be entirely useless if this were not the case. To show that $A(\alpha, x) > 0$, it suffices to show that:

$$\frac{5 + 1/x^2}{1 + x^2} < 4\alpha \tag{90}$$

Put $y = x^2$, so that (90) reduces to

$$y^2 + (\alpha - 5/4)y - 1/4 > 0 \tag{91}$$

This last equation tells us that $A(\alpha, x) > 0$ for all $x$, provided $\alpha > 9/4$.

## A.1 Relative error of the approximation to $\Psi(\alpha, x)$

We have $\Psi(\alpha, x) = A(\alpha, x) + r(\alpha, x)$ and $A(\alpha, x) > 0; r(\alpha, x) > 0$. Now relative error is given by:

$$\begin{aligned} \text{Relative Error} &= \frac{r(\alpha, x)}{A(\alpha, x) + r(\alpha, x)} \\ &< \frac{r(\alpha, x)}{A(\alpha, x)} \end{aligned}$$

Thus, using eq (89), we have:

$$\text{Relative Error} < \frac{\left(35x^4 + 14x^2 + 3\right)}{8\alpha^3 x^5 \left(1 + x^2\right)^4 A(\alpha, x)} \tag{92}$$

Using the expression for $A(\alpha, x)$ from (87), eq (92) becomes:

$$\begin{aligned} \text{Relative Error} &= \frac{35x^4 + 14x^2 + 3}{2\alpha x^2 \left(1 + x^2\right) \left(2\alpha x^4 + (2\alpha - 5)\, x^2 - 1\right)} \\ &< \frac{\left(35x^4 + 14x^2 + 3\right)}{2\alpha x^4 \left(1 + x^2\right) (2\alpha - 5)} \end{aligned} \tag{93}$$

This last inequality holds if

$$2\alpha x^4 > 1 \tag{94}$$

Assuming (94) to be satisfied, proceed to:

$$\text{Relative Error} < \frac{35x^4 + 14x^2 + 3}{(2\alpha - 5)^2 x^4} \tag{95}$$

Now restrict $x$ to $x > (2\alpha - 5)^{-1/4}$, which is compatible with (94). Then, observing that $\left(35x^4 + 14x^2 + 3\right)/x^4$ is a decreasing function of $x$, eq (95) then yields:

$$\text{Relative Error} < \frac{35\,(2\alpha - 5)^{-1} + 14\,(2\alpha - 5)^{-1/2} + 3}{(2\alpha - 5)^2\,(2\alpha - 5)^{-1}} \tag{96}$$

Writing $\epsilon = (2\alpha - 5)^{-1/4}$, we have, for $x > \epsilon$:

$$\text{Relative Error} < \epsilon^4\left(3 + 14\epsilon^2 + 35\epsilon^4\right) \tag{97}$$

From the line immediately preceding inequality (93), we see that, for $x \geq 1$, a much better bound may be obtained. Since the decreasing function $\left(35x^4 + 14x^2 + 3\right)/x^4$ for $x \geq 1$ is greatest at $x = 1$. That is, for $x \geq 1$, we have:

$$\begin{aligned}\text{Relative Error} &< \frac{1}{2\alpha x^2\left(1 + x^2\right)}\frac{35x^4 + 14x^2 + 3}{2\alpha x^4} \\ &\leq \frac{1}{2\alpha x^2\left(1 + x^2\right)}\frac{52}{2\alpha}\end{aligned}$$

Thus,

$$\text{Relative Error} < \frac{13}{\alpha^2 x^2\left(1 + x^2\right)}; \qquad \text{for } x \geq 1 \tag{98}$$

# B Evaluation of $Y(\alpha, x)$

Here, we consider the function $Y(\alpha, x)$, defined by the following integral.

$$Y(\alpha, x) = \int_0^x e^{-\alpha u^2}\left(1 + u^2\right)^{-2} du \tag{99}$$

Write $z = \alpha u^2$ and $\lambda = \alpha x^2$ so that $0 \leq z \leq \alpha x^2 \leq \alpha$. With this change of variable, eq (99) becomes

$$Y(\alpha, x) = \frac{1}{2\sqrt{\alpha}}\int_0^\lambda e^{-z}\left(1 + \frac{z}{\alpha}\right)^{-2} z^{-1/2} dz \tag{100}$$

A mean value theorem of the differential calculus enables us to write

$$\left(1 + \frac{z}{\alpha}\right)^{-2} = 1 - \frac{2z}{\alpha}\left(1 + \frac{\xi}{\alpha}\right)^{-3} \tag{101}$$

where $0 < \xi < z \leq \alpha$. Using eq (101) in eq (100), we have:

$$2\sqrt{\alpha}Y(\alpha, x) = \int_0^\lambda z^{-1/2} e^{-z} dz - \frac{2}{\alpha}\int_0^\lambda z^{1/2}\left(1 + \frac{\xi}{\alpha}\right)^{-3} e^{-z} dz$$

so that, writing

$$A(\lambda) = \int_0^\lambda z^{-1/2} e^{-z} dz \tag{102}$$

and

$$R(\lambda) = \frac{2}{\alpha} \int_0^\lambda z^{1/2} \left(1 + \frac{\xi}{\alpha}\right)^{-3} e^{-z} dz \tag{103}$$

we have:

$$2\sqrt{\alpha} Y(\alpha, x) = A(\lambda) - R(\lambda) \tag{104}$$

where $A(\lambda)$ is the estimate for $2\sqrt{\alpha}Y(\alpha, x)$ with remainder term (or error) $R(\lambda)$. Note that $A(\lambda) \geq 0$ and $R(\lambda) \geq 0$.

Now, since $\xi < \alpha$, we have:

$$0 < R(\lambda) < \frac{2}{\alpha} \int_0^\lambda z^{1/2} e^{-z} \tag{105}$$

Integrating by parts once gives:

$$\frac{\alpha}{2} R(\lambda) < -\sqrt{\lambda} e^{-\lambda} + \frac{1}{2} A(\lambda)$$

This implies the existence of a function $\Box(\lambda)$ such that $\Box(\lambda) > 0$ and

$$R(\lambda) = \frac{2}{\alpha} \left( \frac{1}{2} A(\lambda) - \sqrt{\lambda} e^{-\lambda} - \Box(\lambda) \right) \tag{106}$$

Therefore, the relative error of the approximation satisfies the following

$$\frac{R(\lambda)}{A(\lambda) - R(\lambda)} < \frac{\frac{2}{\alpha}\left(\frac{1}{2}A(\lambda) - \sqrt{\lambda}e^{-\lambda}\right)}{A(\lambda) - \frac{2}{\alpha}\left(\frac{1}{2}A(\lambda) - \sqrt{\lambda}e^{-\lambda} - \Box(\lambda)\right)}$$

$$= \frac{A(\lambda) - 2\sqrt{\lambda}e^{-\lambda}}{(\alpha - 1) A(\lambda) + 2\sqrt{\lambda}e^{-\lambda} + 2\Box(\lambda)}$$

Finally:

$$\text{Relative Error} = \frac{R(\lambda)}{A(\lambda) - R(\lambda)} < \frac{1}{\alpha - 1} \tag{107}$$

As for the estimate, $A(\lambda)$, we have:

$$A(\lambda) = \int_0^\lambda z^{-1/2} e^{-z} dz = \int_0^{\sqrt{\lambda}} 2e^{-t^2} dt = \sqrt{\pi}\, \text{erf}\, \sqrt{\lambda} \tag{108}$$

Therefore,

$$Y(\alpha, x) = \frac{1}{2\sqrt{\alpha}} \left( \sqrt{\pi}\, \text{erf}\left(x\sqrt{\alpha}\right) - R(\lambda) \right)$$

or

$$Y(\alpha, x) = \frac{1}{2\sqrt{\alpha}} \sqrt{\pi}\, \text{erf}\left(x\sqrt{\alpha}\right) \left(1 - \frac{R(\lambda)}{A(\lambda)}\right)$$

But, since $R(\lambda) > 0$, eq (107) implies that:

$$0 < \frac{R(\lambda)}{A(\lambda)} < \frac{1}{\alpha - 1}$$

then, finally:

$$Y(\alpha, x) = \frac{\sqrt{\pi}}{2\sqrt{\alpha}} \operatorname{erf}\left(x\sqrt{\alpha}\right)(1-\rho) \tag{109}$$

where

$$0 < \rho < \frac{1}{\alpha - 1} \tag{110}$$

# C Estimation of $K(\alpha)$ for $\alpha \gg 1$

We wish to estimate $K(\alpha)$ for $\alpha \gg 1$. From eq (18) we have:

$$K(\alpha) = \frac{\pi e^{\alpha}}{4}(1-2\alpha)\left(1 - \operatorname{erf}\sqrt{\alpha}\right) + \frac{\sqrt{\pi\alpha}}{2} \tag{111}$$

and from [6, sec 7.1.23, 7.1.24], we have

$$1 - \operatorname{erf}\sqrt{\alpha} = \frac{e^{-\alpha}}{\sqrt{\pi\alpha}}\left(1 - \frac{1}{2\alpha} + \eta(\alpha)\right) \tag{112}$$

where $0 < \eta(\alpha) < 3/4\alpha^2$. Using the expression (112) for $1 - \operatorname{erf}\sqrt{\alpha}$ in (111), it is seen that

$$K(\alpha) = \frac{\sqrt{\pi}}{2\sqrt{\alpha}}\left(1 - \frac{1}{4\alpha} - \left(\alpha - \frac{1}{2}\right)\eta(\alpha)\right) \tag{113}$$

We may therefore write

$$K(\alpha) = \frac{\sqrt{\pi}}{2\sqrt{\alpha}}\left(1 + \eta^*(\alpha)\right) \tag{114}$$

where

$$\eta^*(\alpha) = -\frac{1}{4\alpha} - \left(\alpha - \frac{1}{2}\right)\eta(\alpha) \tag{115}$$

and $\eta^*(\alpha) < 0$ if $\alpha > \frac{1}{2}$. Also, for $\alpha > 1/2$, we have

$$|\eta^*(\alpha)| < \frac{1}{\alpha} - \frac{3}{8\alpha^2} \tag{116}$$

In particular, we see that $|\eta^*(\alpha)| < 1/\alpha$. From eq (114):

$$K(\alpha) = \frac{\sqrt{\pi}}{2\sqrt{\alpha}}\left(1 + \eta^*(\alpha)\right) \tag{117}$$

Therefore, if we consider $\sqrt{\pi}\eta^*(\alpha)/2\sqrt{\alpha}$ as the error term, then the relative error of the approximation $\sqrt{\pi}/2\sqrt{\alpha}$ to $K(\alpha)$ is given by:

$$\text{Relative Error} = \left|\frac{\eta^*(\alpha)}{1 + \eta^*(\alpha)}\right| = \frac{-\eta^*(\alpha)}{1 + \eta^*(\alpha)} \tag{118}$$

since $-1 < -1/\alpha < \eta^*(\alpha) < 0$. Now, on $[-1/\alpha, 0)$, the decreasing function $f$, where $f(x) = -x/(1+x)$, attains its maximum when $x = -1/\alpha$; hence

$$\text{Relative Error} < \frac{1/\alpha}{1 - 1/\alpha} = \frac{1}{\alpha - 1} \tag{119}$$