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The article is not written primarily from the geometrical point of view, but emphasis is placed on the role that spreadsheets can play in developing an environment in which ideas can be explored and discovered by students. The benefits of using Excel in the context of a geometric investigation are outlined in terms of student control and ease of learning of the environment together with the spreadsheet's ability to cope with exceptions without halting.

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October 5, 2005

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The use of Excel to explore old, well-known geometrical theorems is set in the context of Napoleon's Theorem. A number of geometrical ideas are drawn into the investigation, and spreadsheet examples are given to show how Excel can effectively model such concepts. The article concludes with a generalisation with respect to lines drawn from the vertices of a triangle.

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1 Introduction

As noted in Baker and Sugden [3], the use of spreadsheets in secondary school mathematics has great potential. In particular, Wood and D'Souza [14], Hughes-Hallett [10] and Friedlander [8] stress its relevance to the learning of algebraic concepts, while Beare [4] stresses the support that spreadsheets can give to a range of learning styles. In this paper, I look at a particular part of the Year 11-12 curriculum in mathematics, where students are introduced to matrices and vectors, and at the same time to the powerful matrix capabilities of Excel. Thus, topics such as solving sets of linear equations, Markov chains with transition matrices and the use of Leslie matrices [11] for exploring population sustainability are all set in the context of spreadsheets that allow the calculations of matrix multiplication and inversion to be undertaken without the heavy workload of calculation by hand, but with understanding. As a result, it is realistic to encourage students to develop interactive spreadsheets that display the results of matrix operations in a graphical form, as recommended by Houghton [9]:

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For those ready for this level of abstraction, spreadsheets provided an interactive environment with numerous opportunities for further manipulation. Such manipulation is an important early step in preparation for higher levels of scientific visualization. For example, changes to the raw data are immediately reflected in the graph. Further, the spreadsheet's numerous options for graphing provide immediate access to a wide range of perspectives on a given set of data.

2 Geometry and Spreadsheets

Recently, a number of authors [1], [2], [12] have demonstrated the value of using spreadsheets as a means of displaying geometrical figures and relationships. Through the use of rotation matrices and graphs of geometric objects, underlying geometric patterns and relationships can be quickly displayed and explored.

Always on the lookout for situations that call for graphical, rather than algebraic approaches, the opening chapters of *Mathematical Encounters of the 2nd kind* [7] provided me with the inspiration to use Excel to explore Napoleon's Theorem. This theorem, which is fully explained at Bogomolny's cut-the-knot site [6], can be readily demonstrated with an interactive Excel spreadsheet that the students can prepare for themselves with little guidance, once some initial ground rules for creating geometric objects in Excel have been explored. The Excel Skill Sheet, *Drawing a Triangle*, which I use with the class, is available from the SiE http://www.sie.bond.edu.au/index.htm, and it leads students through the stages of charting a triangle whose shape is controlled by three scrollbars. The position of A is given by formulae such as `=2*COS(theta_a)` and `=2*SIN(theta_a)` and similar formulae give the positions of B and C. When the value of theta_a is controlled by a scrollbar, the locus of A is a circle of radius 2 and in this way, triangle ABC can be given any desired shape and orientation (see Figure 1).

3 Napoleon's Theorem

Once the students have achieved mastery of triangle-drawing, they are ready to move on to Napoleon's Theorem:

On each side of a triangle, let equilateral triangles be constructed that are external to it. The centres of the equilateral triangles form an equilateral triangle.

Given that vectors and matrices are current topics when this theorem is introduced, we address the problem of drawing the outwardly equilateral triangles by discussing how vectors and matrices can be used to create the diagram for Napoleon's Theorem in Excel. For example, to locate the point D, which is the third point of the equilateral triangle on side BC, we use the following formula for BC and similar formulae for CA and AB.

$$\begin{pmatrix} d_x \\ d_y \end{pmatrix} = \begin{pmatrix} \cos(-\pi/3) & -\sin(-\pi/3) \\ \sin(-\pi/3) & \cos(-\pi/3) \end{pmatrix} \left(\begin{pmatrix} c_x \\ c_y \end{pmatrix} - \begin{pmatrix} b_x \\ b_y \end{pmatrix} \right) + \begin{pmatrix} b_x \\ b_y \end{pmatrix} \tag{1}$$

To see these formulae in action, the reader is asked to open the spreadsheet *Napoleon and Beyond* and select the first worksheet, *Napoleon.* The worksheet shows how the

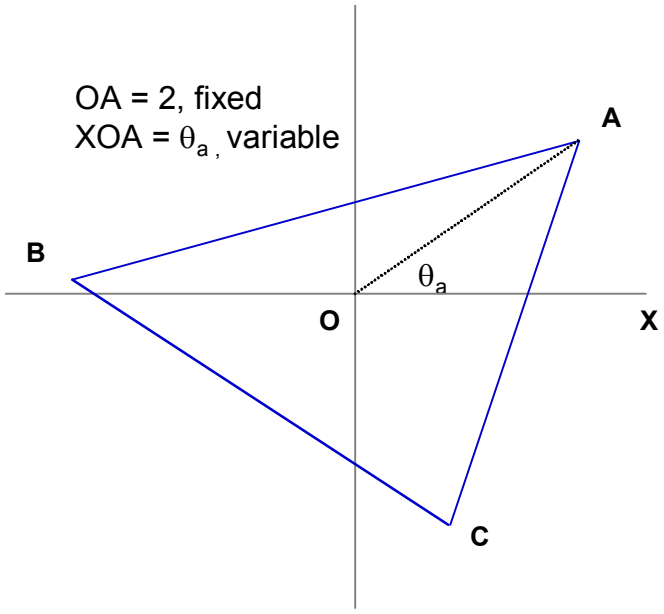

Figure 1: Drawing a Triangle in Excel

above formula is applied. It is probably important to say that, once the skill sheet on triangle drawing has been mastered, students need no further help to produce a similar spreadsheet, apart from adding the check feature given below the chart. The check feature shows that, no matter what shape triangle ABC is given, the sides of PQR are always equal in length.

There is no claim here that the chart which demonstrates Napoleon's Theorem represents any sort of proof that the theorem is true. But it certainly does seem to confirm the result. This confirmation certainly helps to maintain the students' motivation when, later in the course, we use complex numbers to construct an algebraic argument that settles its veracity. The complex number approach is one of many suggested by Bogomolny [**?**], who gives a very comprehensive coverage of this elegant theorem.

Note too that the spreadsheet contains a minor addition to the theorem; the students are asked to add in the lines AP, BQ and CR as in Figure 2, and to comment on what they notice. It escapes no-one's attention that these three lines are concurrent. We return to this addition later in the paper.

4 An Extension of Napoleon's Theorem

Davis [7, pages 6-11] also discusses two very similar theorems, which he struggled with. The first of these is given as follows:

Equilateral triangles are constructed on the sides of a given triangle and external to it. Prove that the three lines joining the outer vertex of one of the equilateral triangles

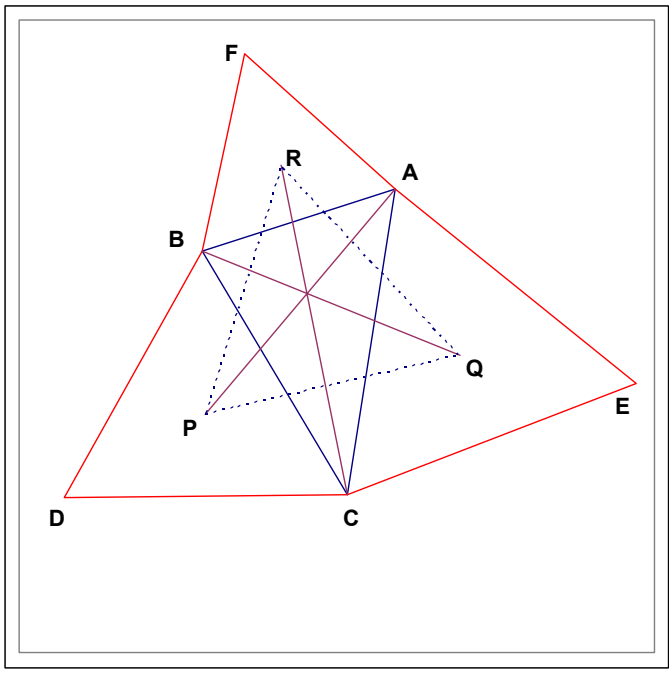

Figure 2: Napoleon's Theorem

	Line AD				
	A	P			AP
x-val	1.175571	-1.04212		Length	3.853553
y-val	1.618034	-1.53343		Slope	1.421058
				Intercept	-0.05252
	B	Q			BE
x-val	-1.96457	2.078171		Length	4.066627
y-val	0.374763	-0.06531		Slope	-0.10885
				Intercept	0.160909
	C	R			CF
x-val	1.071654	-0.7534		Length	4.028643
y-val	-1.68866	1.90288		Slope	-1.9679
				Intercept	0.420255

Figure 3: AD, BE and CF have the same length

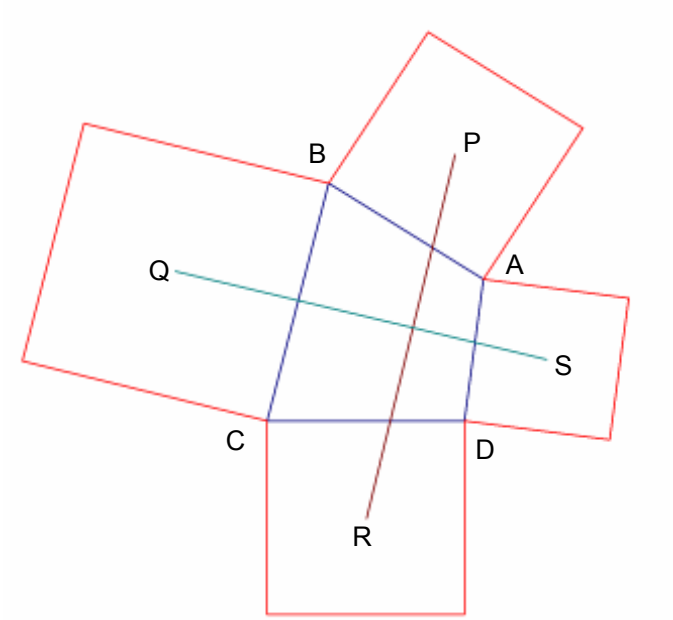

Figure 4: Squares on the sides of a Quadrilateral

to the opposite vertex of the given triangle meet in a point and are equal.

Because the hard work of setting up that chart in Napoleon's Theorem has been done, it takes little time for the students to make a copy of that and edit the copy to produce the worksheet given as *Extension*. Having done this, they find that not only do AD, BE and CF meet, but that, if the Excel calculations are to be believed, these lines are also equal in length, as shown in Figure 3.

The spreadsheet also allows us to reverse the order of A, B and C, making them appear anti-clockwise rather than clockwise. In doing this, the equilateral triangles are drawn not on the outside of ABC but on the inside. Indeed, when only B and C are swapped and the triangle reads as ACB in the anti-clockwise direction, two of the equilateral triangles appear on the inside. When this happens, we find that the lines AD, BE and CF do not always meet. But that is only because they need to be extended. A careful watch of the lengths of these lines shows that they continue to have the same length no matter what order A, B and C are in.

The benefits of creating a good looking spreadsheet, complete with named cells, show up clearly as the students can see how little extra time is needed to demonstrate the extension in spreadsheet form. And the flexibility of the spreadsheet indicates that the more general form of the extension would be to replace the requirement that the equilateral triangles be drawn outside the given triangle with a requirement that, as we travel round the triangle, A to B to C and back to A, the equilateral triangles are always drawn to the left or right of the direction of motion—the phrase frequently used

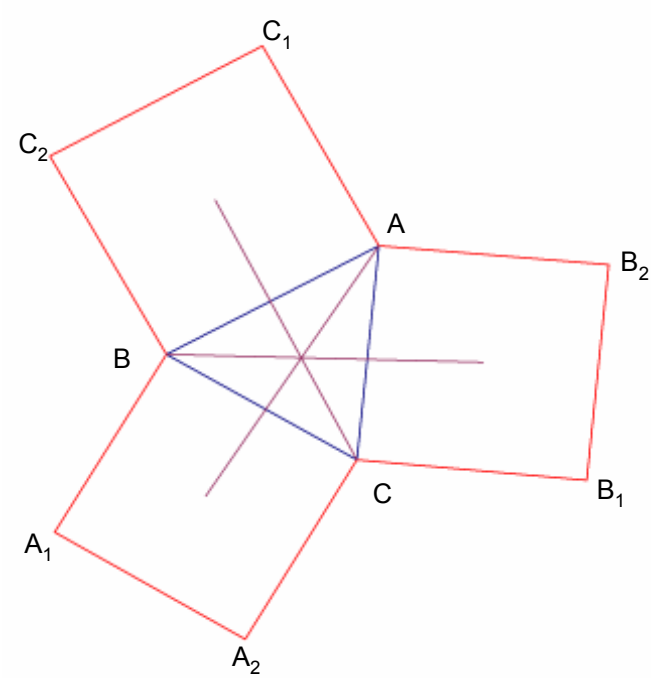

Figure 5: Triangle with Squares

to describe this is *in the same sense*. Thus the extension could be restated as:

Equilateral triangles are constructed in the same sense on the sides of a given triangle. Prove that the three lines joining the third vertex of one of the equilateral triangles to the opposite vertex of the given triangle meet in a point and are equal.

The same applies to Napoleon's Theorem, where the triangles can be consistently on the inside or outside of the given triangle.

5 The Quadrilateral Equivalent

Whilst looking for further examples that could be used with the students, I came across the following challenge on the Wilson website [13].

Given any quadrilateral, construct a square and locate its center on each side of the quadrilateral. Explore the relationship of the two segments defined by connecting the centers of squares on the opposite sides of the quadrilateral.

Wilson's challenge is illustrated in Figure 4, and the parallel between it and Napoleon's Theorem is clear; add a regular four-sided figure to each side of a general four-sided figure. But the results appear to be rather different.

As before, the students were challenged to model the question in an Excel chart. Rather than restrict ourselves to moving the vertices of the quadrilateral around a circle (as was done for the triangle of Napoleon's Theorem) we agreed that only the two points

A and B should move, and that we could leave C and D fixed. After all, we were not out to prove anything, only to use the spreadsheet to investigate some ideas.

The worksheet, *Quadrilaterals*, shows how the Wilson conditions can be animated, and it includes a check that PR and QS (the joins of opposing centres of squares) are both equal and perpendicular.

Because of the facility in the model to fully control the positions of vertices A and B, we can also discover that when these two points coincide, the quadrilateral reduces to a triangle, which leads to the following theorem[1]:

Let squares be constructed on the sides of a given triangle and external to it. The line joining one vertex of the triangle to the centre of the opposite square is perpendicular and equal to the line joining the centres of the two other squares.

6 Triangles with Squares

The above theorem has been given as a special case of Wilson's challenge, when two vertices of the quadrilateral coincide. However, if the spirit of Napoleon's Theorem and its extension is followed, the investigation from the point of view of triangles ought to take a different tack. Thus the students were invited to develop the *Triangles with Squares* worksheet. In this, instead of looking at the relationship between the vertex-centre and centre-centre lines, we turned our attention to the joins of each vertex to the centre of its opposite square, as shown in Figure 5. The striking feature is that the joins of vertices to the centres of opposite squares again meet at a common point.

7 A Generalisation

We now have a growing body of evidence that is pointing in one direction—that there is something special about locating a point in relation to one side of a triangle and then joining that point to the opposite vertex. *Generalisation*, the last worksheet to be developed, is based on the following theorem.

In a triangle ABC, let $\lambda \in \Re$ and let lines of length λBC, λCA and λAB be drawn perpendicular to BC, CA and AB from their respective mid-points, A', B' and C', to the points D, E and F. The lines AD, BE and CF meet at a common point, P.

With very little alteration again, the spreadsheet that models this theorem was derived from *Triangles with Squares*. The only major change is the introduction of a fourth scrollbar to control the value of λ. By changing λ, the lengths of $A'D$, $B'E$ and $C'F$ are all altered, but the lines AD, BE and CF still meet at a point. In order to cover a wide range of values for λ, the position of their meet, called P, is also calculated, and P is added to AD, so that in effect Excel draws the lines ADP, BEP and CFP. This has been done to allow for the situations where AD needs to be extended to meet the other lines, i.e. when P does not lie between A and D.

[1] A spreadsheet for this theorem was prepared by a student at Kimberley College, Carbrook, and as a reward I offered that, should the theorem ever be named, then it should be called Chan's Theorem.

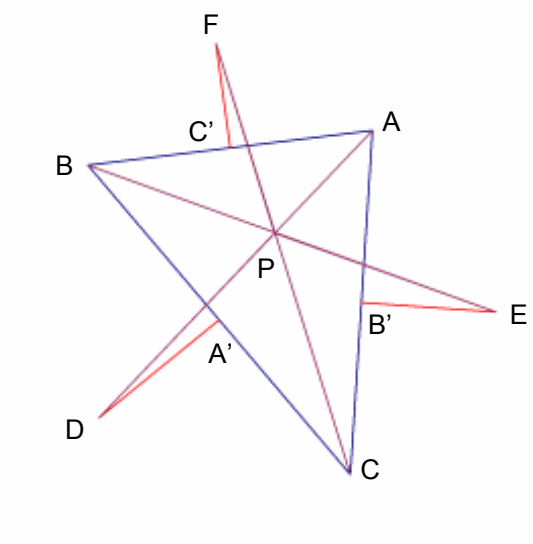

Figure 6: A Generalisation

The *generalisation* is another way of stating Kiepert's Theorem which is also discussed by Bogomolny [5], where instead of the parameter controlling the length of a perpendicular to the sides of the triangle ABC, congruent isosceles triangles are drawn on the sides; the effect is the same even if the articulations are not.

For students, the value of the generalisation is that it links a number of what might be thought of as distinct results. For example, students can see that if we set $\lambda = 0$, we have the median property, that the joins of the vertices of a triangle to the middle of the opposite sides all meet at a point, the centroid. As λ tends to $\pm\infty$, the lines AD, BE and CF approximate the altitudes of ABC, which also meet at a point, the orthocentre. When $\lambda = \sqrt{3}/2$ and the extension to Napoleon's Theorem is obtained. We mentioned earlier that joining the vertices of the Napoleon triangle to the opposite vertices of the original triangle produces lines that meet at a point. This case is covered by $\lambda = 1/(2\sqrt{3})$ and the point at which AD, BE and CF meet is known as the *Fermat point,* which has the property that $PA + PB + PC$ is minimised. Finally, when $\lambda = 1/2$ the same result as that covered in the Triangle with Squares worksheet is demonstrated.

To explore the locus of P, the reader is invited to make two changes to the *Generalisation* worksheet:

1. Right click the scrollbar for λ (which overlaps cell `N2`) and set the Maximum to 4000.

2. Select cell `M2` and change the formula to make $-20 \leq \lambda \leq 20$. The formula should be `=-20 + N2/100.`

Watch how the position of P changes with λ and, in particular, look for points that

P passes through. The locus of P is known as Kiepert's hyperbola and is discussed on the MathWorld site at mathworld.wolfram.com/KiepertHyperbola.html.

8 Conclusions

The aim of this paper has been to show examples of using Excel to explore geometric concepts. Starting from a simple spreadsheet that Grade 11 students are well capable of mastering, *Drawing a Triangle*, the method of generalisation has been to use simple extensions, such as:

- add in the line that joins two points,
- draw a line perpendicular to this line, passing through a given point,
- find the intersection of two lines.

Overall, the outcome of the investigations has been to find some generalisations to known results as well as finding a generalisation that covers a wide range of such results. But we need to stand back a little from the geometric content, and look at the value that has been gained from using Excel in such a context.

Three benefits of using Excel over a conventional programming environment to make these investigations are that:

1. Students have full control over what is drawn on the Excel chart, and are able to construct their own understanding of the generalisation being explored.
2. The requirement for mastering a language's syntax is minimal. Only the formulae employed in drawing lines and the use of names need to be known to be able to carry out the investigation.
3. The reader who followed the investigation of the locus of P will have noticed that at no stage does the spreadsheet collapse because of a `#DIV/0!` problem. This can be a difficult event to trap in a programming language, but it is not an issue when using Excel.

As has been shown, key concepts such as vectors and matrices can be demonstrated in Excel in a way that other media may not allow. Certainly graphics calculators would be inadequate for such demonstrations, and while this author is not fully familiar with geometry-specific environments such as Cabri, it is doubted whether they would be able to do a great deal better ... and much of the calculation would be hidden from the student, who has to make everything happen in Excel.

The five worksheets given have a common look-feel, and were derived one from the other once the basic triangle-drawing component had been established. Thus the labour of producing the worksheets is not too large for students to achieve.

For this author, however, there is one further feature that is considered most important. The charts all have an interactive quality in as much as all key variables can be

controlled by the user, who by simply changing a scrollbar, can completely alter the look of the chart without losing the underlying connections between the points that have been plotted and joined on the charts. . It was because of this interactivity that questions such as "what if we joined ..." were supported, and thanks to which, new ideas have been explored by students in an environment that is wholly under their control.

9 Acknowledgement

A final version of this paper was read by Dr A. Bogomolny who pointed to the classical form of the Generalisation, Kiepert's Theorem, and to an explanation of the locus of the point P. My sincere thanks are due to Dr. Bogomolny for kindly bringing these matters to my attention.

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Appendix: Proof of the Generalisation

Let the points of the diagram be represented by complex numbers, such that, for example:

$$A' = \frac{\mathbf{b}+\mathbf{c}}{2} \tag{2}$$

$$D = \lambda i\,(\mathbf{b}-\mathbf{c}) + \frac{\mathbf{b}+\mathbf{c}}{2} = \left(\frac{1}{2}+\lambda i\right)\mathbf{b} + \left(\frac{1}{2}-\lambda i\right)\mathbf{c} \tag{3}$$

where $\mathbf{b} = b_x + ib_y$ and $\mathbf{c} = c_x + ic_y$.

The representation of D uses the property that multiplication by i is equivalent to rotation by $\pi/2$ in an anticlockwise direction.

Consider the point P, which is defined as the point where AD and BE meet. We shall find a condition for P to lie on both AD and BE, and from that condition show that P also lies on CF, which will complete the proof.

Given that P lies on AD, a parameter α can be defined to give the position of P as:

$$\mathbf{p} = \mathbf{a} + \alpha\,(\mathbf{d}-\mathbf{a}) \tag{4}$$

$$= \alpha\left(\left(\frac{1}{\alpha}-1\right)\mathbf{a} + \left(\frac{1}{2}+\lambda i\right)\mathbf{b} + \left(\frac{1}{2}-\lambda i\right)\mathbf{c}\right) \tag{5}$$

The coordinate system is now chosen such that P lies as the origin. Thus:

$$0 = \left(\frac{1}{\alpha}-1\right)\mathbf{a} + \left(\frac{1}{2}+\lambda i\right)\mathbf{b} + \left(\frac{1}{2}-\lambda i\right)\mathbf{c} \tag{6}$$

Therefore:

$$2\left(\frac{1}{\alpha}-1\right)a_x + b_x - 2\lambda b_y + c_x + 2\lambda c_y = 0 \tag{7}$$

and

$$2\left(\frac{1}{\alpha}-1\right)a_y + 2\lambda b_x + b_y - 2\lambda c_x + c_y = 0 \tag{8}$$

Eliminating the bracketed expression in α from the above equations allows the condition for P to be at the origin to be written as:

$$a_y b_x - 2\lambda a_y b_y + a_y c_x + 2\lambda a_y c_y = 2\lambda a_x b_x + a_x b_y - 2\lambda a_x c_x + a_x c_y \tag{9}$$

This equation can be used with cyclic rotation of **a**, **b** and **c** to give the condition that P is at the origin and lies on BE. This condition is:

$$b_y c_x - 2\lambda b_y c_y + b_y a_x + 2\lambda b_y a_y = 2\lambda b_x c_x + b_x c_y - 2\lambda b_x a_x + b_x a_y \tag{10}$$

These two conditions can be added, and when terms in c_x are collected on the right-hand side and terms in c_y are collected on the left-hand side, the following equation results:

$$c_y a_x - 2\lambda c_y a_y + c_y b_x + 2\lambda c_y b_y = 2\lambda c_x a_x + c_x a_y - 2\lambda c_x b_x + c_x b_y \tag{11}$$

This is the condition that the point P lies on CF and is at the origin. Hence AD, BE and CF do meet at the origin of the chosen coordinate system.

Corollary 1 *The locus of the point P, as defined above, is a conic that passes through the three vertices, the centroid and the orthocentre of the given triangle.*

Given that the locus we aim to show is a conic, affine transformations can be used to greatly simplify the algebra. We assume that $B = (0, 0)$, $C = (1, 0)$ and that the position of A is determined by a parameter c at the coordinates $(c, 1)$. In this coordinate system and using the complex number $z = 1/2 + i\lambda$ to locate the positions of D, E and F in the manner given by Bogomolny [**?**], the lines AD and BE respectively are given by:

$$y = \frac{2(1-\lambda)}{2c-1}x + \frac{2c\lambda - 1}{2c-1} \tag{12}$$

$$y = \frac{1 - 2\lambda + 2\lambda c}{1 + c - 2\lambda}x \tag{13}$$

The (x, y) coordinates of P satisfy both of the above, and thus eliminating λ between these two equations will give a locus of P in terms of x and y. The expression obtained on elimination of λ is:

$$\frac{-2x + y(2c-1) + 1}{-x + c} = \frac{-x + y(c+1)}{x(c-1) + y} \tag{14}$$

or

$$x^2(1-2c) + xy\left(2c^2 - 2c\right) + y^2(2c-1) + x(2c-1) + y\left(1 - c - c^2\right) = 0 \tag{15}$$

This locus is of degree 2, which passes through the centroid, $\left(\frac{1}{3}(c+1), \frac{1}{3}\right)$, the orthocentre, $\left(c, c - c^2\right)$, as well as the vertices A, B and C.