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Lost in translation: honest misunderstandings and ex post disputes

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Abstract

We give a formal treatment of optimal risk sharing contracts in the face of ambiguity. The central idea is that boundedly rational individuals do not have access to a language sufficiently rich to describe all possible states of nature. The ambiguity in a contract arises from contractual clauses that are interpreted by the parties in different ways. The cost of ambiguity is represented in terms of dispute costs. Taking the potential for dispute into account, we find that risk averse agents may forgo potential gains from risk sharing and choose incomplete contracts instead.

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1 Introduction

The idea that contractual incompleteness is related in some way to ambiguity seems appealing (Mukerji, 1998). Appropriately enough, however, this idea is itself ambiguous. In ordinary usage, the term 'ambiguity' refers to the semantic property of propositions or sets of propositions admitting multiple, mutually inconsistent interpretations. The fact that the terms of contracts may be interpreted in different ways by different parties is well-known and is a disincentive to the adoption of complex contracts with exotic provisions. This fact is naturally expressed with reference to the potential ambiguity of contracts. Most discussion of the relevant issues of knowledge, awareness and so on is undertaken in terms of the semantic interpretation of propositions, and consistent with this, we will use the term 'semantic ambiguity' to refer to ambiguity as a property of propositions that cannot be expressed in terms of events (measurable subsets of a state space).

On the other hand, in state-contingent approaches to decision theory, the term 'ambiguity' is commonly used in relation to the fruitful body of work beginning with Ellsberg (1961), who built on earlier contributions by Keynes (1921) and Knight (1921). Appropriately enough, there is no generally agreed definition of 'ambiguity' in this context. Broadly speaking, however, events are ambiguous if preferences regarding acts measurable with respect to those events cannot be rationalized by a subjective probability distribution. It seems plausible to suggest that contracting over ambiguous events will prove difficult. This suggestion has been formalized with specific reference to the incomplete contracts literature by Mukerji (1998) and with reference to financial markets by Mukerji and Tallon (2001). We will use the term 'probabilistic ambiguity' to refer to ambiguity as a property of events for which there exists no well-defined probability. Grant and Quiggin (2006) show that semantic ambiguity implies probabilistic ambiguity, but not vice versa.

As argued by Maskin and Tirole (1999), responding to the work of Hart and Moore (1988, 1999), constraints arising from inability to describe, or even foresee possible states of nature, will not, in general, prevent perfectly rational agents from achieving optimal contracts. This insight seems particularly applicable to notions of semantic ambiguity, which reflect the incapacity of the natural languages used by boundedly rational human agents to achieve the unbounded expressive power of formal languages implicit in most models of contracting.

The purpose of this paper is to consider the role of semantic and probabilistic ambiguity in the specification of incomplete contracts. In the process, we will show, using a model of unforeseen contingencies incorporating bounded rationality (Grant and Quiggin 2006; Heifetz, Meier and Schipper 2006), that the semantic and probabilistic notions of ambiguity are closely related. That is, Ellsberg's choice of terminology was indeed apposite.

Unlike most of the economics literature on incomplete contracts, we shall not consider strategic misrepresentations or other 'opportunistic' behavior of one or more parties to a contract. Rather disputes, if they arise, will do so from 'honest misunderstandings' arising from the inherent ambiguity of natural language. Furthermore, we shall assume that each individual has complete information relative to the representation of the world available to them in the absence of contracting. Hence the only source of ambiguity in our model will be from individuals' different interpretations of the world.

The paper is organized as follows. We begin with a simple example designed to motivate the analysis. Next we describe the situation in which contracting takes place, adapting the model of Grant and Quiggin (2006). The key idea is that the parties have access to different state-space representations of the world, even though they share the same natural language and use it to describe contingencies. The different models used by the parties may be viewed as alternative coarsenings of the fully-specified state-space description of the world that would be available to an unboundedly rational observer.

We then consider risk-sharing contracts which involve a rule specifying a state-contingent transfer vector. The main interest of the paper is in the specification of the sharing rule in the case when the state space cannot be described fully and unambiguously. The 'external' perspective of an unboundedly rational observer may be used to characterize the constrained-optimal contract between the parties, and to derive conditions under which such a contract may be reached, even though the parties themselves do not have access to the external description. Using this contract as a benchmark, we consider the roles of ambiguity and risk aversion in determining the extent of contracting and risk-sharing.

2 Motivating example

In informal discussions of ambiguous contracts, it is common to refer to 'gray areas'. Some contracts, or contingencies specified in contracts, are seen as having gray areas, thereby giving rise to possibilities of disagreement and dispute, while others are seen as relatively clear-cut and unambiguous.

We develop these ideas in an example, specified as follows.[1] Suppose two individuals *Row* (Rowena) and *Col* (Colin) are contemplating entering into a risk-sharing contract. They will draw a card from a pack. It may be white at both ends (ranging in shades of light gray towards the middle), black at both ends (ranging in shades of dark gray towards the middle), or white at one end and black at the other (ranging in shades of light gray in one half to darker shades of gray in the other half). If both ends are white (black) the card is deemed 'white' ('black'). If one end is white and the other is black, the card ranges in shade from white-to-black or black-to-white.

Each player sees the world as black or white, and resolves shades of gray to whichever is nearer. However, *Row* always observes the top half of the card, while *Col* always observes the bottom half. Thus, if the card ranges from black at the top to white at the bottom, *Row* will observe shades of dark gray, which she will construe as black, while *Col* will observe shades of light gray, which he will construe as white. The underlying state space and the two individuals' partitions of the black–white spectrum are summarized in the following table:

		Col's observation	
	Card drawn is:	*white at bottom*	*black at bottom*
Row's observation	*white at top*	white, white	white,black
	black at top	black,white	black,black

Suppose the state-contingent endowments of the two individuals are given in the following bi-

[1] We are indebted to Bob Brito for this suggestion.

matrix,

		Col's endowment	
	Card drawn is:	*white at bottom*	*black at bottom*
Row's endowment	*white at top*	0 1	1 1
	black at top	0 0	1 0

The bottom-left-hand entries of each cell of the bi-matrix are constant within a row while the top-right-hand entries are constant within a column. That is, each individual faces a single source of uncertainty that is measurable with respect to his own partition of the state space.

We assume that both players are risk-averse and view the two elements of their respective partitions as 'exchangeable' (Chew and Sagi 2006). Hence both parties would prefer the risk-free endowment yielding 1/2 in every state. So, ignoring (for the moment) any possibility of future disagreement and dispute, both would find it attractive to sign a risk-sharing contract comprising the following contingent transfer from *Col* to *Row* (expressed in terms of their common natural language):

$$t = \begin{cases} -1/2 & \text{if the card drawn is `white'} \\ 1/2 & \text{if the card drawn is `black'} \end{cases} .$$

In the formal framework developed below, if such a contract were signed, the presumption is that each party translates the contingencies on which the transfer function t depends into her or his own formal language. For *Row*, this entails interpreting 'the card drawn is white' as meaning 'top is white', while for *Col*, this entails interpreting 'the card drawn is black' as meaning 'bottom is black'.

The card shading from black at the top to white at the bottom creates a possibility for disagreement since *Row* will observe dark gray shading, interpret this as 'black', and so believe that she is entitled to receive a payment. *Col* will in the same situation observe light gray shading, interpret this as 'white', so he will also expect a payment. Hence, a disagreement will ensue. The card shading from white at the top to black at the bottom also is inconsistent with the explicit contingencies stated in the contract but since in this state both were expecting to make a transfer to the other, we presume that this can be resolved amicably with the 'surplus' shared between the parties.

Boundedly rational players, in this setup, are unable (in the absence of some increase in effort) to formulate a state description sufficiently refined to encompass this possibility, allowing the contract to specify a resolution. However, they may nonetheless be aware (in a sense made precise by Grant and Quiggin 2006) that disputes are possible. Depending on the weight they place on this possibility, they may choose a contract which offers only partial hedging, or even no contract at all. This corresponds closely to the risk–uncertainty distinction of Knight (1921) whose main concern was with uncertainties that could not be hedged through market contracts such as insurance, and therefore reduced to manageable risk. Uncertainty of this kind was central to Knight's idea of entrepreneurship.

Some results are intuitively apparent. The parties will benefit less from a hedging contract the larger (as observed from an external perspective unavailable to them) is the gray area giving rise to dispute. They will benefit more from a hedging contract the more risk-averse they are, that is, the stronger their preference for the non-stochastic endowment over the original endowment. Thus risk and ambiguity work in opposite directions. The aim of the present paper is to develop a formal model within which these propositions can be assessed.

3 Objective world and natural languages

Following Grant and Quiggin (2006), we start with a formal language and an external description of the world, in which each state of the world is a complete description of the truth or falsity of each primitive proposition. More precisely, given a finite non-empty set of primitive propositions $P = \{p_1, ..., p_K\}$, a state of the world $\omega = (\omega_1, ..., \omega_K) \in \{0, 1\}^K$ is a K-dimensional vector where for all $p_k \in P$, $\omega_k = 0$ if and only if proposition k is false. A state of the world is a complete description of the truth and/or falsity of each primitive proposition. The set of states of the world is denoted by $\Omega = 2^K$. From the perspective of an unboundedly rational external observer, there is a probability measure f on Ω.

3.1 Individual logical languages

Individuals in our model will not, in general, be in a position to formulate or check all the propositions in P, and will therefore be unable by themselves to give an exhaustive specification of the state space. We shall assume, however, that their individual logical languages, if combined and closed under standard logical operations, would generate an exhaustive specification of Ω, with expressive power equivalent to P.

Let $I = \{1, 2\}$ be the player set. Each player $i \in I$ can only check the truth or falsity of some non-empty subset of primitive propositions which we denote by $P^i \subseteq P$ and for which $P^1 \cup P^2 = P$. Adopting the natural order from P for the propositions in P^i, a state of the world $\omega = (\omega_1, ..., \omega_K) \in \{0, 1\}^K$ induces a unique P^i-dimensional vector $s^i(\omega) = (s^i_{p_k})_{p_k \in P^i} \in \{0, 1\}^{P^i}$ where $s^i_{p_k} = 0$ if and only if $\omega_k = 0$. We adopt $S^i := \{0, 1\}^{P^i}$ as the *personal state space* of player i, and presume that he observes the *personal state* $s^i(\omega) \in S^i$ when the state $\omega \in \Omega$ obtains.

Restating this point from the semantic perspective, we do not presume that a player knows the state space Ω or the actual state of the world. Rather, each of the players $i = 1, 2$ has access to a state-contingent representation S^i of the world, which may be viewed as a coarsening of Ω, as in Grant and Quiggin (2006). More precisely, when ω obtains, player i observes $s^i = s^i(\omega)$.

A simple distinction between our formulation and a more standard one is noted by comparing ours to that of Osborne and Rubinstein's formulation of a Bayesian Game (1994, Section 2.6). In both formulations, the starting point is a set of states of the world and a signal function. In our formulation, player i upon receiving $s^i \in S^i$ knows only the truth and falsity of the propositions in P^i. He may be unaware of the propositions in $P - P^i$ and of the set of states Ω. In the formulation of a Bayesian game by Osbourne and Rubinstein, the player knows also the set $\{\omega \in \Omega : s^i(\omega) = s^i\}$, by which we can interpret him as knowing all the other propositions in $P - P^i$ as well as the set Ω.

We now turn our attention to the logical language of a player. We presume that this can be described by a set of formulas L^i obtained inductively from P^i as follows:

F1: any $p \in P^i$ is a formula;

F2: if A and B are formula, so are $\neg A$,$\neg B$, $A \wedge B$, and $A \vee B$ (where $\neg$, $\wedge$ and $\vee$, refer to the logical operations of 'NOT', '(inclusive) OR' and 'AND', respectively);and

F3: Every formula is obtained by a finite number of applications of F1 and F2.

From a semantic point of view, the truth or falsity of each proposition A in a player i's language L^i is determined for each personal state $s^i(\omega) \in S^i$ from the truth or falsity of A at the corresponding $\omega \in \Omega$. The *truth assignment* of player i is the function $k^i : S^i \times P^i \longrightarrow \{0,1\}$ defined by $k^i(s^i, p) = s^i_p$ for each $p \in P^i$.We then extend the function k^i to the remaining formula L^i inductively as follows. For any $A, B \in L^i$:

T1: $k^i(s^i, \neg A) = 1$ iff $k^i(s^i, A) = 0$;

T2: $k^i(s^i, A \wedge B) = 1$ iff $k^i(s^i, A) = 1$ and $k^i(s^i, B) = 1$; and

T3: $k^i(s^i, A \vee B) = 1$ iff $k^i(s^i, A) = 1$ or $k^i(s^i, B) = 1$.

In summary, a state of the world ω determines a signal $s^i(\omega)$ that player i receives. The truth or falsity of a formula $A \in L^i$ at state $s^i(\omega)$ is then determined by k^i and T1, T2, T3.

For example, suppose that $P = \{p_1, p_2, p_3\}$, and $P^i = \{p_1, p_3\}$. Then,
$\Omega = \{(0,0,0),(0,0,1),(0,1,0),(0,1,1),(1,0,0),(1,0,1),(1,1,0),(1,1,1)\}$, and by our convention, $S^i = \{(0,0),(0,1),(1,0),(1,1)\}$. If the state of the world $\omega = (\omega_1, \omega_2, \omega_3)$ occurs, then player i receives the signal $s^i(\omega) = (\omega_1, \omega_3)$. For example, if $\omega = (0,1,0)$, then player i observes $s^i(\omega) = (0,0)$. In this case, only proposition p_2 is true, but player i only sees that p_1 and p_3 are false. He might not even conceive of p_2 since it is not in his language. He can, however, check the truth or falsity of any formula in L^i using his signal and T1 to T3.

Interactions between players with limited awareness gives rise to a lattice structure similar to that described by Heifetz, Meier and Schipper (2006).[2] In this lattice structure, languages may be ordered by the subset relationship on the set of primitive propositions from which they are generated, or equivalently by the refinement of the partition of Ω that they generate. The meet and join relationships $\wedge$ are defined in the natural way. For the purposes of the present

[2] For a similar development in a dynamic context of individual unawareness, see Grant and Quiggin (2006).

paper we can focus on the minimal non-trivial lattice consisting of $L^1, L^2, L^1 \vee L^2, L^1 \wedge L^2$. We do not model contingencies unforeseen by either party, which would require a language richer than $L^1 \vee L^2$. Conversely, all contingencies expressible in $L^1 \wedge L^2$ are mutually expressible (this term will be made precise below) so there is no need to consider any less refined language than $L^1 \wedge L^2$.

3.2 Probabilities, beliefs and awareness

We presume that player i's prior belief g^i over S^i is consistent with the probability measure f that describes the beliefs of an external observer in the following sense. For each $s^i \in S^i$, let $\Omega(s^i) = \{\omega \in \Omega : s^i(\omega) = s^i\}$. Then we require that:

$$g^i\left(s^i\right) = \sum_{\omega \in \Omega(s^i)} f(\omega).$$

For any random variable $\mathbf{X}$ measurable with respect to S^i, define

$$\mathrm{E}_{S^i}[\mathbf{X}] = \sum_{s^i \in S^i} [\mathbf{X}(s)]\, g^i(s^i).$$

Similarly, for any random variable Y measurable with respect to Ω, define

$$\mathrm{E}_{\Omega}[\mathbf{X}] = \sum_{\omega \in \Omega} [\mathbf{Y}(\omega)]\, f(\omega).$$

In the terminology of Grant and Quiggin (2006), the individual's beliefs, in the pre-contract situation, represent Bayesianism in a restricted domain. That is, neither party's beliefs about the propositions accessible to them in the natural language are conditioned on implicit beliefs about propositions in $P - P^i$, so their beliefs may be represented simply as the marginal distribution derived from f. Thus, the model presented here represents a minimal departure from the standard assumption of unbounded rationality.

Although players in this model have probabilistic beliefs, they are not in a position to act as consistent Bayesians. They may encounter evidence (such as a contract dispute) that reveals that their representation of the world is incomplete. Normally parties to a contract will have past experience of contracts some of which have ended in dispute. Thus they will be aware (in a sense made precise by Grant and Quiggin (2006) of the possibility of disputes, even though this possibility is not explicitly represented in their state-contingent model of the world.

4 Contracts

We consider choice settings in which players can write contracts to share risk via transfers. In the absence of contracts we presume that each player i faces risk described by a personal state-contingent income vector $Z^i \in \mathbb{R}^{S^i}$. When the state $s^i \in S^i$ obtains, player i receives the endowment $Z^i(s^i)$. He is presumed to be an expected utility maximizer with utility function $u^i : \mathbb{R} \to \mathbb{R}$. The expected utility of his personal-state contingent income vector is his reservation utility which we denote by $\bar{u}^i = \sum_{s^i \in S^i} g^i\left(s^i\right) \times u^i(Z^i\left(s^i\right))$.

In addition to the propositions in P which can be used to describe the world, we assume that the players speak in a shared informal 'natural' language. This is described by a finite set of *contingencies* $Q = \{q_1, ..., q_M\}$. Contracts will be written in terms of these contingencies. A vector $x = (x_q)_{q \in Q} \in \{0,1\}^Q$ is interpreted as meaning that all the contingencies $q \in Q$ with $x_q = 1$ are true and that all those with $x_q = 0$ are false. We will allow players to write transfers on the vectors in $\{0,1\}^Q$.

A *contract* is a function $t : \{0,1\}^Q \to \mathbb{R}$.

The function t is a transfer function specifying the amount of transfer from player 2 to player 1 as a function of the contingencies that are true.

Each player interprets the contract using his formal language. For this purpose we assume that each player i has a *translation function* which is an injection $T^i : Q \to L^i$. He uses his translation function to determine the truth or falsity of each contingency. Formally, each player $i \in I$ checks the vector $\hat{x}^i\left(s^i\right) \in \{0,1\}^Q$, where $\hat{x}^i_q(s^i) = 1$ if and only if $k^i(s^i, T^i(q)) = 1$. For notational convenience we define $x^i(\omega) \equiv \hat{x}^i \circ s^i(\omega)$. When a contract t is in place and state ω obtains, player i expects the transfer from player 2 to player 1 to be $t(x^i(\omega))$. Since the players have different languages and translation functions, there is no guarantee that $t(x^1(\omega)) = t(x^2(\omega))$ for each $\omega \in \Omega$. We do impose, however, one condition on the translation functions. We presume that the translation functions T^1 and T^2 are *consistent on mutually expressible parts* in the sense that for each $q \in Q$ and for all i, j,

$$T^i(q) \in L^j \text{ implies that } T^i(q) = T^j(q).$$

If the translation functions are consistent on mutually expressible parts and $P^1 = P^2$ $(= P)$, then $T^1 = T^2$ everywhere, since in this case $L^1 = L^2$. In such a case, there would be no chance for disputes in our setup. The possibility of a dispute might occur only when $T^1(q) \neq T^2(q)$. Restricting analysis to translation functions that are consistent on mutually expressible parts means that a dispute might only occur when one player's translation of some contingency q is not expressible in the other player's language. More generally, we shall refer to a contingency $q \in Q$ as an *unambiguous contingency* if it is semantically equivalent for the two parties, that is, $x_q^1(\omega) = x_q^2(\omega)$ for each $\omega \in \Omega$. Let $\bar{Q} \subset Q$, denote the set of unambiguous contingencies.

The *support* of a contract t is the set $Q_t \subseteq Q$ of contingencies that make a difference to the transfer function t. We write Q_t to emphasize that this set depends on the transfer function t. More precisely, $q \in Q_t$ iff for some $x \in \{0,1\}^Q$, $t(x) \neq t((1 - x_q)_q, x_{-q})$ where x_{-q} is the vector obtained by deleting the q-th element of x $[\equiv (x_q, x_{-q})]$, and thus $t\left((1 - x_q)_q, x_{-q}\right)$ is the transfer made in the event that the truth value of q is reversed relative to x. Hence a contract t is an *unambiguous contract* if $Q_t \subseteq \bar{Q}$. Players will never have any problem writing contracts on this set of contingencies.

4.1 Misunderstandings and dispute costs

Although various reasons why contracts might be incomplete have been suggested (such as that checking the truth value of contingencies is costly to the parties), we focus in this paper on the potential cost of disputes arising from different interpretations of a contract. When a contract involves contingencies that are translated in different ways by the two parties, there is a potential for disputes to arise over which contingencies have actually been met even after the checking has been done by each party. These disputes will in general be costly, so forward-looking parties might choose incomplete contracts in an attempt to avoid them.

If a state $\omega \in \Omega$ is realized and $t\left(x^1(\omega)\right) - t\left(x^2(\omega)\right) < 0$, we assume that although there is a misunderstanding in terms of the amount of the transfer, this can be amicably resolved. However, if $t\left(x^1(\omega)\right) - t\left(x^2(\omega)\right) > 0$, that is, one or both parties think they should receive a payment greater than the other party thinks it should make, then a dispute occurs. Dispute costs in any state ω are assumed to depend on the extent of disagreement. More precisely, the size of a dispute at a state ω is defined by

$$d(\omega; t) = t\left(x^1(\omega)\right) - t\left(x^2(\omega)\right).$$

and we suppose that the costs of a dispute are increasing in the size of the dispute $d(\omega; t)$, that is, we suppose there are (convex and increasing) functions $\phi^1 : \mathbb{R} \to \mathbb{R}$, $\phi^2 : \mathbb{R} \to \mathbb{R}$. We assume $\phi^i(0) = 0, i = 1, 2$, that is, if the players agree, there is no dispute. The surplus-splitting assumption for $d < 0$ is embodied in a requirement that $\phi^1(d) + \phi^2(d) = d$, $d < 0$. Since ϕ^i is convex, it has right and left derivatives everywhere. Denote the left-hand and right-hand derivatives by $\left(\phi^i_+\right)'(d)$ and $\left(\phi^i_-\right)'(d)$, respectively. Our conditions imply that $\phi^1(d)$ and $\phi^1(d)$ are linear for $d \leq 0$. We might expect a kink at $d = 0$ but do not preclude kinks at $d > 0$.

From the external perspective, we define the expected utility for i conditional on $\left(t, s^i\right)$ by:

$$CEU^i(t, s^i) = \frac{\sum_{\omega \in s^i} u^i\left(\mathbf{Z}^i\left(s^i\right) + (-1)^{(i-1)} t\left(x\left(s^i\right)\right) - \phi^i\left(d(\omega; t)\right)\right) f(\omega)}{g^i(s^i)}. \quad (1)$$

Now consider the preferences of individual i, who does not have access to a description sufficiently refined to express the terms on the RHS of (1). We assume that the these preferences over contracts may be represented by a real-valued function $U^i : \{0, 1\}^Q \to \mathbb{R}$. We assume that, for every unambiguous contract t:

$$\begin{aligned} U^i(t) &= \sum_{s^i \in S^i} g^i(s^i) \times \left[CEU(t, s^i)\right] \\ &= \sum_{s^i \in S^i} u^i\left(\mathbf{Z}^i\left(s^i\right) + (-1)^{(i-1)} t\left(\hat{x}^i\left(s^i\right)\right)\right) g^i(s^i). \end{aligned} \quad (2)$$

That is, in the absence of ambiguity, each individual's preferences coincide with those derived from the external perspective. This is natural, since, for unambiguous contracts, both individuals have access to a common language expressive enough to compute $CEU^i(t, s^i)$, given the available information.

Definition 1 *An individual is* ambiguity-neutral *if for every contract t:*

$$\begin{aligned} U^i(t) &= \sum_{s^i \in S^i} g^i(s^i) \times \left[CEU(t, s^i)\right] \\ &= \sum_{\omega \in \Omega} u^i\left(\mathbf{Z}^i\left(s^i(\omega)\right) + (-1)^{(i-1)} t\left(x^i(\omega)\right) - \phi^i\left(d(\omega; t)\right)\right) f(\omega). \end{aligned} \quad (3)$$

If this condition is satisfied, the individual's preferences coincide with those given by $CEU^i(t, s^i)$, even in ambiguous situations. Note that this equality does not imply that the individual has access to the values $f(\omega)$ and $\phi^i(d(\omega; t)$ required for the computation of $CEU^i(t, s^i)$. It simply states that an ambiguity-neutral individual's ranking of possible contracts coincides with that which would be computed by the external observer, given knowledge of the utility index u^i. This point is of crucial relevance in relation to the analysis of Maskin and Tirole (1999). Even if individual preferences are ambiguity-neutral, individuals cannot make calculations with respect to a probability distribution over outcomes and therefore cannot write and implement the outcome-contingent contracts required to achieve the first-best. In particular, although individuals are assumed to be aware (in the sense of Grant and Quiggin 2006) of the possibility of dispute, they do not have access either to a state-contingent description of the world fine enough to include the dispute states (since these states arise precisely from coarseness in individual's partitions of the state space) or to a probability distribution including the probability of dispute.

More generally, we may define ambiguity aversion in a fashion consistent with the characterization of risk aversion, proposed by Yaari (1969), as being more averse to moves from certainty to risk than a risk-neutral individual. Supposing that the individual never prefers an ambiguous contract when the external planner would prefer an unambiguous one, they may be characterized as ambiguity-averse. More formally.

Definition 2 *An individual is* ambiguity-averse *if, for any t, t' such that t is unambiguous and t' is ambiguous,*

$$\sum_{s^i \in S^i} g^i(s^i) \times \left[CEU(t, s^i)\right] \geq \sum_{s^i \in S^i} g^i(s^i) \times \left[CEU(t', s^i)\right] \Rightarrow U^i(t) \geq U^i(t').$$

That is, an individual is regarded as being ambiguity-averse if he ranks an unambiguous contract over an ambiguous one whenever an external observer would rank the unambiguous contract more highly. Note that, as would be expected, ambiguity neutrality is a polar case of ambiguity aversion where the converse implication holds. To define the opposite polar case, we will say that a player is *maximally ambiguity-averse* if $U^i(t) \geq U^i(t')$ for any t, t' such that t is unambiguous and t' is ambiguous.

If the individual is risk-neutral as well as being ambiguity-neutral (that is, $u^i(\cdot)$ is affine) then $U^i(t)$ reduces to

$$\mathrm{E}_{S^i}\left[\mathbf{Z}^i(s^i) + (-1)^{(i-1)}\, t\left(\hat{x}^i\right)\right] - \Phi^i(t)$$

where

$$\Phi^i(t) = \sum_{\omega\in\Omega} \phi^i\left(d(\omega;t)\right) f(\omega),$$

is the unconditional expected dispute cost. That is, the utility of the contract is simply the expected value of the contract were it unambiguous, less the expected dispute cost.

4.2 The Nash bargaining contract

As is standard in economics, we shall assume the parties select the contract that corresponds to the Nash bargaining solution. Here, the individual rationality of each player i is evaluated relative to his state-contigent expected utility $\bar{u}^i$ which he would obtain in the absence of a contract.

The Nash bargaining solution may be obtained by choosing a transfer function $t:\{0,1\}^Q \to \mathbb{R}$ to maximize

$$W = \sum_{i=1}^{2} \ln\left(U^i(t) - \bar{u}^i\right).$$

4.3 Optimal contracts

In order to provide a benchmark for assessing the welfare characteristics of contracts potentially available to the players, we now consider the problem faced by an external planner seeking to maximize welfare under a range of constraints on the set of feasible contracts. The lattice structure gives rise to a range of such solution concepts.

4.3.1 Constrained optimal contract

First let us consider the constrained-optimal contract, evaluated from the external perspective. The cost of a dispute at $\omega\in\Omega$ can be abbreviated by

$$\begin{aligned}\phi^i(\omega) &= \phi^i(t\left(x^1(\omega)\right) - t\left(x^2(\omega)\right)) \\ &= \phi^i((-1)^{(i-1)}\, t\left(x^i(\omega)\right) + (-1)^i\, t\left(x^{(3-i)}(\omega)\right)).\end{aligned}$$

The planner chooses a transfer function $t : \{0,1\}^Q \to \mathbb{R}$ to maximize:

$$W^{\mathrm{P}} = \sum_{i=1}^{2} \ln \Delta u^i$$

where

$$\Delta u^i = \sum_{\omega \in \Omega} u^i \left(\mathbf{Z}^i \left(s^i(\omega) \right) + (-1)^{(i-1)} t \left(x^i(\omega) \right) - \phi^i(\omega) \right) f(\omega) - \bar{u}^i.$$

For each $\omega \in \Omega$, set $u^i(\omega) := u^i \left(\mathbf{Z}^i \left(s^i(\omega) \right) + (-1)^{(i-1)} t \left(x^i(\omega) \right) - \phi^i(\omega) \right)$, and for each $x \in \{0,1\}^Q$, set $\Omega^i(x) := \left\{ \omega : x^i(\omega) = x \right\}$.

Since our structural assumptions have ensured the planner's problem is a concave program, the first order conditions are both necessary and sufficient.

The first order conditions consist of two conditions for each x:

$$\begin{aligned} & \sum_{i=1}^{2} \frac{1}{\Delta u^i} \left[\sum_{\omega \in \Omega^i(x)} (-1)^{(i-1)} \left(u^i \right)' (\omega) f(\omega) \right. \\ + & \sum_{\omega \in \Omega^1(x) \cap [\Omega \setminus \Omega^2(x)]} (-1)^i \left(u^i \right)' (\omega) \left(\phi^i_+ \right)' (\omega) f(\omega) \\ + & \left. \sum_{\omega \in \Omega^2(x) \cap [\Omega \setminus \Omega^1(x)]} (-1)^{(i-1)} \left(u^i \right)' (\omega) \left(\phi^i_- \right)' (\omega) f(\omega) \right] \leq 0; \text{ and} \end{aligned} \tag{4}$$

$$\begin{aligned} & \sum_{i=1}^{2} \frac{1}{\Delta u^i} \left[\sum_{\omega \in \Omega^i(x)} (-1)^{(i-1)} \left(u^i \right)' (\omega) f(\omega) \right. \\ + & \sum_{\omega \in \Omega^1(x) \cap [\Omega \setminus \Omega^2(x)]} (-1)^i \left(u^i \right)' (\omega) \left(\phi^i_- \right)' (\omega) f(\omega) \\ + & \left. \sum_{\omega \in \Omega^2(x) \cap [\Omega \setminus \Omega^1(x)]} (-1)^{(i-1)} \left(u^i \right)' (\omega) \left(\phi^i_+ \right)' (\omega) f(\omega) \right] \geq 0. \end{aligned} \tag{5}$$

The first condition (4) comes from the consideration of raising the transfer t at x. The second condition (5) comes from the consideration of lowering the transfer t at x. The first term in the brackets of both conditions corresponds to the situation where the player i sees x. The second term corresponds to the dispute event when player 1 sees x and player 2 sees something else. The third term corresponds to the dispute event when player 2 sees x and player 1 sees something else.

Notice that if a particular truth assignment of the contingencies x does not lead to any disputes, then the first-order-condition on $t\left(x\right)$ collapses to:

$$\sum_{i=1}^{2}\frac{(-1)^{(i-1)}\left[\sum_{s^i:x(s^i)=x}\left(u^i\right)'\left(\mathbf{Z}^i\left(s^i\right)+(-1)^{(i-1)}\,t\left(x\right)\right)g^i\left(s^i\right)\right]}{\Delta u^i}=0. \tag{6}$$

Notice that the constrained-optimal contract is the one that would be chosen by an external planner seeking to maximize the product of gains in expected utility in the knowledge that the players would have to implement the contract and deal with any resulting disputes. This contract is available to the players as a solution to their bargaining problem. However, the constrained-optimal contract might not be chosen by the players bargaining in the absence of a planner. Recall that the ranking of contracts by individual i is given by $U^i(t)$ which might not coincide with the ranking $\sum_{s^i\in S^i} g^i(s^i)\times\left[CEU(t,s^i)\right]$ used by the planner.

We can specify a sufficient condition for the two contracts to coincide. When both players are ambiguity neutral, (3) implies:

Lemma 1 *For ambiguity-neutral players, the planner's constrained-optimal contract coincides with the Nash bargaining solution for the players.*

4.3.2 Unconstrained-optimal contract

The unconstrained-optimal contract can be viewed as the contract that would be written by an external planner with access to the maximally expressive formal language $L^1\vee L^2$ to maximize the Nash product again using $\sum_{s^i\in S^i} g^i(s^i)\times\left[CEU(t,s^i)\right]$ in place of $U^i(t)$. Equivalently, this is the optimal contract conditional on Ω.

Unless the players have access to $L^1\vee L^2$, the unconstrained optimal contract will not be a feasible solution to the contracting problem facing them. In the special case where $P^1=P^2=P=\{T^1(q):q\in Q\}$, so that $x^1(\omega)=x^2(\omega)=\omega$ for all $\omega\in\Omega$, the parties can implement the unconstrained optimal contract without any possibility of dispute. Hence, the optimal constrained and optimal unconstrained contracts coincide.

In this solution, the first-order-condition on $t(\omega)$ becomes, for each ω:

$$\sum_{i=1}^{2} \frac{(-1)^{(i-1)} \left(u^i\right)' \left(\mathbf{Z}^i \left(s^i(\omega)\right) + (-1)^{(i-1)} t(\omega)\right)}{\left(\sum_{\omega \in \Omega} u^i \left(\mathbf{Z}^i \left(s^i(\omega)\right) + (-1)^{(i-1)} t(\omega)\right) f(\omega) - \bar{u}^i\right)} = 0.$$

4.3.3 Optimal unambiguous contract

Finally, consider the optimal unambiguous contract. We restrict contracts to those whose support are unambiguous, that is, $Q_t \subseteq \bar{Q}$, and define for each x, the equivalence class

$$[x] = \left\{x' : x'_q = x_q \;\forall q \in \bar{Q}\right\}.$$

For a contract t to be unambiguous, we require $t(x') = t(x)$, for all x' in $[x]$.

In this solution, the planner chooses transfer function $t : \{0,1\}^Q \to \mathbb{R}$ with support in $\bar{Q}$, to maximize

$$W^{\text{U}} = \sum_{i=1}^{2} \ln \left[\sum_{s^i:\, \hat{x}^i(s^i)=x} u^i \left(\mathbf{Z}^i \left(s^i\right) + (-1)^{(i-1)} t(x)\right) g^i \left(s^i\right) - \bar{u}^i \right].$$

This yields the first-order-condition on $t(x)$,

$$\sum_{i=1}^{2} \frac{(-1)^{(i-1)} \left[\sum_{\left\{s^i \in (\hat{x}^i)^{-1}[x]\right\}} \left(u^i\right)' \left(\mathbf{Z}^i \left(s^i\right) + (-1)^{(i-1)} t(x)\right) g^i \left(s^i\right) \right]}{\Delta u^i} = 0, \tag{7}$$

where $\left(\hat{x}^i\right)^{-1}[x]$ is the set of signal realizations for player i that map into $[x]$.

Unlike the unconstrained optimal contract, the optimal unambiguous contract can be implemented by the parties. In the situation where dispute costs ϕ are large enough to rule out any ambiguous contract, the set of feasible contracts consists solely of unambiguous contracts. In this case, it is easy to see that the optimal unambiguous solution is the same as the Nash bargaining solution. We record this fact as a lemma.

Lemma 2 *The optimal unambiguous contract is the Nash bargaining solution for players restricted to the set of unambiguous contracts.*

In the special case where $P^1 = P^2 = P = \{T^1(q) : q \in Q\}$, consistency on mutually expressible parts ensures that the optimal unambiguous contract coincides with the constrained and unconstrained optimal contracts.

4.4 Welfare

We now consider a welfare ranking of the solutions discussed above. We focus on the case where the problem is symmetric.

Definition 3 *A contracting problem is symmetric if for any* t*, there exists* t' *such that* $U^i(t) = U^{3-i}(t')$*. Furthermore, if* $U^i(t) = U^{3-i}(t)$*,*
then $\mathrm{E}\Omega\left[u^i\left(\mathbf{Z}^i + (-1)^{(i-1)}t - \phi^i\right)\right] = \mathrm{E}\Omega\left[u^{3-i}\left(\mathbf{Z}^{3-i} + (-1)^i t - \phi^{3-i}\right)\right]$*,* $i = 1, 2$*.*

The first part of the definition requires that for each available contract there exists a counterpart in which the utility outcomes are reversed. The second part requires that if a contract yields the same (*ex ante*) utility for both players then it also yields the same expected utility for the two players calculated from the perspective of the external observer.

Notice that in a symmetric problem, the external planner's constrained-optimal solution, the players' Nash bargaining solution and the optimal unambiguous contract are all ones for which $U^1(t) = U^2(t)$ and thus they also result in the same expected utility for both players calculated from the perspective of the external observer. Our main result is:

Proposition 3 *In a symmetric problem, with ambiguity-averse players the solutions may be Pareto-ranked by the external observer (in decreasing order) as follows:*

(i) *Constrained-optimal solution;*

(ii) *Nash bargaining solution; and*

(iii) *Optimal unambiguous contract.*

In terms of the players' own preferences $\left(U^1, U^2\right)$*, (ii) is Pareto-dominant and the ranking between (i) and (iii) depends on ambiguity aversion: (ii)=(i) for ambiguity-neutral players and (ii)=(iii) for maximally ambiguity-averse players.*

Proof. First, in terms of the external observer, we have: (a) since the Nash bargaining solution is available to the external observer, it can be no better than the solution to the constrained problem, and hence $(i) \geq (ii)$; and (b) if $(iii) > (ii)$, then ambiguity averse players must also rank (*iii*) above (*ii*), but this cannot hold since the optimal unambiguous contract is available to the players in their Nash bargaining problem, hence it follows that (*ii*) $\geq$ (*iii*), as required. Second, in terms of the players themselves, we have, by construction, (*ii*) as Pareto-dominant in the set of equal utility contracts, so (*ii*) $\geq$ (*i*) and (*ii*) $\geq$ (*iii*). Lemma 1 implies (*ii*) = (*i*) for ambiguity-neutral players and lemma 2 implies (*ii*) = (*iii*) for maximally ambiguity-averse players. ■

4.5 *Ex ante* refinement

The solutions available to the players through contracting are Pareto-dominated by the unconstrained optimal solution that would be selected and implemented by an external planner. Although this fact cannot be expressed in the language available to the players, they will in general be aware that the contracts available to them are incomplete and subject to the possibility of dispute. Expressing awareness of such possibilities requires an extension of the language available to the players, such as that defined by Grant and Quiggin (2006). As Grant and Quiggin show, beliefs about such possibilities will not, in general be expressible in probabilistic terms, and, in particular, will not admit of contingent contracting.

Suppose that the players are aware of the possibility of dispute and are also aware that by expending effort to improve their mutual understanding before contracting, they can reduce or eliminate the ambiguity in the contracts available to them in their common language. We will refer to the result of such effort as *ex ante refinement* of the state space. The cost of effort may be assumed known, but the benefits available from refinement of the state space will not be. Nevertheless, the players may choose to incur the cost in the hope that it will be less than the resulting benefits.

From the external perspective, it is possible to compute the expected benefits and therefore determine whether they exceed the costs. Some instances are illustrated in the examples below.

In the *ex ante* position, the players will not be able to make such calculations on a probabilistic basis. In making their decisions, they may rely on past experience, as is modelled, for example, in the case-based decision theory of Gilboa and Schmeidler (1995). Alternatively, they may rely on the advice of external advisers whose understanding is closer to that of the ideal external planner invoked here as an analytical device.

4.6 Example – Incomplete contracting and incomplete risk-sharing

We now develop the example presented informally in Section 2. We assume the players are ambiguity-neutral so the constrained-optimal and Nash bargaining solutions coincide.

The set of propositions that determine the states of the world are $P = \{p_1, p_2\}$, where the propositions are interpreted as $p_1 =$ 'The card is white at the top' and $p_2 =$ 'The card is white at the bottom'. The set P generates the underlying state space $\Omega = \{(1,1), (1,0), (0,1), (0,0)\}$, where, for example, $(1,0)$ corresponds to the proposition $p_1 \wedge \neg p_2$ and the state in which the card ranges in shades of gray from white at the top to black at the bottom. The probability over the state space is given by: $f(1,1) = f(0,0) = (1-\varepsilon)/2$ and $f(0,1) = f(1,0) = \varepsilon/2$, $\varepsilon \in (0, 1/2)$. Observe that the number ε represents the probability of obtaining a 'gray' card.

Row has access to the proposition p_1, that is $P^{Row} = \{p_1\}$ and *Col* has access to the set $P^{Col} = \{p_2\}$. The state spaces for *Row* and *Col* are given by $S^{Row} = \{(1)^{Row}, (0)^{Row}\}$ and $S^{Col} = \{(1)^{Col}, (0)^{Col}\}$. Hence $g^i(s^i) = 1/2$, for $s^i = 1, 0$, and for $i = Row, Col$.

The state-contingent endowments of the two parties are given by: $Z^{Row}(s^{Row}) = s^{Row}$ and $Z^{Col}(s^{Col}) = 1 - s^{Col}$. That is, a card perceived to be 'white' by *Row* is good for her, and a card perceived to be 'black' by *Col* is good for him.

To describe a contract in our framework, we need the set of contingencies in the natural language. We start with $Q = \{w\}$ where w corresponds to the natural language contingency 'the card drawn is white'. Formally, we have the translation functions for *Row* and *Col* defined by: $T^{Row}(w) = p_1$ and $T^{Col}(w) = p_2$.

Parties are risk-averse and preferences over lotteries conform to expected utility with a concave and strictly increasing utility preference scaling function, $u(\cdot)$. Let the cost of dispute for both players be given by the twice-continuously-differentiable-everywhere-except-at-zero function $\phi : \mathbb{R} \to \mathbb{R}$, with $\phi(d) = d/2$, for $d \leq 0$, $\phi'_+(0) > 1/2$ and $\phi''(d) > 0$, for $d > 0$. The rationale for us to require $\phi'(d) > 1/2$ for $d > 0$, is so that if the size of dispute increases by 1 the additional cost summed across both parties is more than 1.

Note that this is a symmetric problem, and the Nash bargaining solution will also be symmetric. The class of symmetric contracts involving partial hedging with transfers based on (w) consists of contracts t, where $t(x) = \delta(1/2 - x)$, $\delta \in [0,1]$, recalling that $x = 1$ corresponds to w is true. Notice $\delta = 0$ corresponds to no contract, leaving both parties fully exposed to the risk associated with the color of the card drawn. By contrast, $\delta = 1$ corresponds to a 'full' risk-sharing contract.

The constrained-optimal value of δ depends on the parameters of the model: u, ϕ and ε. To evaluate the constrained-optimal δ, first note the *ex ante* 'expected' utility for ambiguity-neutral players of

$$U^i(t) = \left(\frac{1-\varepsilon}{2}\right) u\left(1 - \frac{\delta}{2}\right) + \frac{\varepsilon}{2} u(1) + \frac{\varepsilon}{2} u\left(\frac{\delta}{2} - \phi(\delta)\right) + \left(\frac{1-\varepsilon}{2}\right) u\left(\frac{\delta}{2}\right)$$

Differentiating with respect to δ yields the first order necessary condition (for an interior solution, that is $\delta \in (0,1)$):

$$-\frac{(1-\varepsilon)}{4} u'\left(1 - \frac{\delta}{2}\right) - \frac{\varepsilon}{2} u'\left(\frac{\delta}{2} - \phi(\delta)\right)\left(\phi'(\delta) - \frac{1}{2}\right) + \left(\frac{1-\varepsilon}{4}\right) u'\left(\frac{\delta}{2}\right) = 0$$
$$\Rightarrow \frac{(1-\varepsilon)}{2}\left[u'\left(\frac{\delta}{2}\right) - u'\left(1 - \frac{\delta}{2}\right)\right] = \varepsilon u'\left(\frac{\delta}{2} - \phi(\delta)\right)\left(\phi'(\delta) - \frac{1}{2}\right). \quad (8)$$

We observe that $\delta^* = 1$ is optimal iff $\varepsilon = 0$, since as $\delta \to 1$, the left-hand side of (8) goes to zero, while the right-hand side is strictly positive unless $\varepsilon = 0$.

For $\delta^* = 0$ to be an optimum (i.e., for the contract to leave the two parties completely exposed to the risk associated with the color of the card drawn) we require

$$\frac{(1-\varepsilon)}{2}\left[u'(0) - u'(1)\right] < \varepsilon u'(0)\left(\phi'_+(0) - \frac{1}{2}\right).$$

Rearranging yields

$$\frac{1}{2} \times \frac{u'(0) - u'(1)}{u'(0)} < \frac{\varepsilon}{1-\varepsilon}\left(\phi'_+(0) - \frac{1}{2}\right). \quad (9)$$

That is, if inequality (9) holds then the constrained-optimal contract is incomplete in the sense that it leaves both individuals fully exposed to the risk associated with the color of the card. The inequality (9) may be interpreted as saying that the marginal benefit of reducing the unhedged risk is less than the marginal cost of dispute incurred by conditioning payments on a proposition open to ambiguous interpretation. Intuitively, this condition is more likely to hold: (i) the more likely a dispute will arise (that is, the greater is ε), (ii) the larger is the marginal cost of disagreement, given by. $\phi'_+(0)$; and (iii) the less risk averse the individual is.

To see (iii), first notice that inequality (9) always holds for a risk neutral person since the left-hand side is zero and the right-hand side is positive (unless $\varepsilon = 0$). More generally, suppose v is less risk averse than u, that is, there exists a strictly increasing and convex function ψ, such that $v = \psi \circ u$. Then we have,

$$\begin{aligned} \frac{v'(0) - v'(1)}{v'(0)} &= \frac{\psi'(u(0))(u'(0)) - \psi'(u(1))(u'(1))}{\psi'(u(0))(u'(0))} \\ &\leq \frac{\psi'(u(0))(u'(0)) - \psi'(u(0))(u'(1))}{\psi'(u(0))(u'(0))} \\ &= \frac{u'(0) - u'(1)}{u'(0)}. \end{aligned}$$

The inequality follows from the fact that the convexity of ψ implies $\psi'(u(1)) \geq \psi'(u(0))$.

4.6.1 'Refining' the state space *ex ante*

Suppose now instead, if both parties each incur a cost today of k they can 'learn' about shades of gray and hence have access to the full state space Ω .

The Nash bargaining contract is now the unconstrained optimal contract. This contract sets $t(\omega_1, \omega_2) = (1 - \omega_1 - \omega_2)/2$, and yields a net expected utility equal to

$$(1-\varepsilon) u \left(\frac{1}{2} - k\right) + \varepsilon \left[\frac{1}{2} u(0-k) + \frac{1}{2} u(1-k)\right].$$

So *Row* and *Col* would benefit from incurring the cost k of learning, if

$$U^i(t) < (1-\varepsilon) u \left(\frac{1}{2} - k\right) - \varepsilon \left[\frac{1}{2} u(0-k) + \frac{1}{2} u(1-k)\right]$$

That is,

$$\begin{aligned} (1-\varepsilon)\left(\left[u\left(\frac{1}{2} + \frac{1-\delta^*}{2}\right) - u\left(\frac{1}{2} - k\right)\right] - \left[u\left(\frac{1}{2} - k\right) - u\left(\frac{1}{2} - \frac{1-\delta^*}{2}\right)\right]\right) \\ +\varepsilon\left([u(1) - u(1-k)] + \left[u\left(0 - \left[\phi(\delta^*) - \frac{\delta^*}{2}\right]\right) - u(0-k)\right]\right) < 0 \quad . \quad (10) \end{aligned}$$

Row and *Col* cannot calculate this value *ex ante,* but it can be derived from the external perspective which is available to them after refinement.

Notice that for an affine (i.e. risk neutral) utility function, inequality (10) holds if

$$k < \varepsilon \left(\phi \left(\delta^*\right) - \delta^*/2\right)/2. \tag{11}$$

Since, unhedged risk is costly for a risk averse person and the marginal utility of incurring the *ex ante* refinement cost is also lower for higher incomes than for the lower incomes, it follows that inequality (11) is a sufficient condition for *Row* and *Col* to benefit from refinement.

5 Concluding comments

In this paper, we have shown how ambiguity about the terms of a contract can create obstacles to the achievement of complete risk sharing. Ambiguity may give rise to disputes even when both parties to a contract act honestly, with no attempt to conceal or misrepresent private information, and even though rational agents would prefer to avoid disputes if possible. Nevertheless, given the limits of precision in language, risk averse economic agents may prefer ambiguous contracts when the constraints imposed by unambiguous contracts do not permit sufficient risk sharing potential. On the other hand, when the potential costs of dispute are large, and risk sharing contracts are ambiguous, optimal risk sharing will not be achieved.

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